DIFFUSION ALONG TRANSITION CHAINS OF INVARIANT TORI AND AUBRY-MATHER SETS

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ABSTRACT. We describe a topological mechanism for the existence of diffusing orbits in a dynamical system satisfying the following assumptions: (i) the phase space contains a normally hyperbolic invariant manifold diffeomorphic to a two-dimensional annulus, (ii) the restriction of the dynamics to the annulus is an area preserving monotone twist map, (iii) the annulus contains sequences of invariant one-dimensional tori that form transition chains, i.e., the unstable manifold of each torus has a topologically transverse intersection with the stable manifold of the next torus in the sequence, (iv) the transition chains of tori are interspersed with gaps created by resonances, (v) within each gap there is prescribed a finite collection of Aubry-Mather sets. Under these assumptions, there exist trajectories that follow the transition chains, cross over the gaps, and follow the Aubry-Mather sets within each gap, in any specified order. This mechanism is related to the Arnold diffusion problem in Hamiltonian systems. In particular, we prove the existence of diffusing trajectories in the large gap problem of Hamiltonian systems. The argument is topological and constructive.

1. INTRODUCTION

In this paper we study a topological mechanism of diffusion and chaotic orbits related to the Arnold diffusion problem. We consider a normally hyperbolic invariant manifold diffeomorphic to a two-dimensional annulus. We assume that the dynamics restricted to the annulus is given by an area preserving monotone twist map. We assume that inside the annulus there exist invariant primary one-dimensional tori (homotopically nontrivial invariant closed curves) with the property that the unstable manifold of each torus has topologically transverse intersections with the stable manifolds of all sufficiently close tori. Sequences of such tori and their heteroclinic connections form transition chains of tori. The successive transition chains of tori are interspersed with gaps. We assume that each gap is a region in the annulus bounded by two invariant primary tori of one of the following types: (i) a Birkhoff Zone of Instability, i.e., it contains no invariant primary torus in its interior (ii) it contains only finitely many invariant primary tori in its interior, such that each torus is either isolated, or it consists of a hyperbolic periodic orbit together with its stable and unstable manifolds that are assumed to coincide. Within each gap we prescribe a finite collection of Aubry-Mather sets. We prove the existence of orbits

that shadow the primary tori in each transition chain, cross over the gaps that separate the successive transition chains, and also shadow the specified Aubry-Mather sets within each gap.

The motivation for this result is the Arnold diffusion problem of Hamiltonian systems. This problem asserts that all sufficiently small perturbations of generic, integrable Hamiltonian systems exhibit orbits along which the action variable changes substantially; also, there exist chaotic orbits that can be coded through symbolic dynamics. A classification of nearly integrable systems proposed in [13] distinguishes between *a priori* stable systems, in which the unperturbed system can be expressed in terms of action-angle variables only, and *a priori* unstable systems, in which the unperturbed system contains both action-angle and hyperbolic variables. In the *a priori* stable case analytical results have been announced in [47]. In the a priori unstable case there have been several analytical results in the last several years (see [13, 54, 51, 52, 11, 18, 3, 12]). Some of the approaches involve variational methods, geometric methods, or topological methods. See [20] for an overview on the Arnold diffusion problem, applications, and additional references.

It is relevant in applications to detect, combine, and compare different mechanisms of diffusion displayed by concrete systems. In many models, as well as in numerical experiments, diffusion can only be observed for perturbations of sizes much larger than those considered by the analytical approaches [14, 40, 39, 36, 28]. For these types of problems, the geometric and topological approaches are particularly advantageous as they yield constructive methods to detect diffusion, quantitative estimates on diffusing orbits, and explicit conditions that can be verified in concrete examples.

In this paper we describe a general method to establish the existence of diffusing orbits for a large class of dynamical systems. The dynamical systems under consideration are not assumed to be small perturbations of integrable Hamiltonians. Moreover, some systems that are not Hamiltonian can be considered. Our method requires the existence of certain geometric objects that organize the dynamics, and employs topological tools to establish the existence of diffusing orbits. The existence of the geometric objects can be verified in concrete systems through analytical methods, or through numerical methods, or through a combination of thereof.

We illustrate our method in the case of a Hamiltonian system consisting of a pendulum and a rotator with a small periodic coupling. We give an analytic argument for the existence of diffusing orbits. In [15] the topological method is applied to show, with the aid of a computer, the existence of diffusing orbits in the spatial restricted three-body problem, where the two primaries are the Sun and the Earth; this model is not nearly integrable. Similar ideas appear in [10, 21, 22].

The diffusing orbits detected by this approach follow transition chains of invariant tori up to the gaps between these transition chains, and then cross the gaps following the inner dynamics restricted to the annulus. The orbits that cross the gaps follow Birkhoff connecting orbits that go from one boundary of the gap to the other, or Mather connecting orbits, that shadow a prescribed sequence of Aubry-Mather set inside the gap, or homoclinic orbits.

Related approaches, that use Birkhoff's ideas to understand generic drift properties of random iterations of exact-symplectic twist maps, appear in [48, 38]. These yield the concept of a polysystem, which is a locally constant skew-product over a

Bernoulli shift [43]. Polysystems are a key ingredient in the approach of the instability problem in *a priori* unstable systems in [6]. Arnold diffusion-type orbits for random iterations of flows of families of Tonelli Hamiltonians are studied in [42], based on an extension of the pseudograph method from [3].

The diffusing orbits that we obtain in this paper are similar to those found through variational methods, as in [11, 12]; they are however not necessarily action minimizing. Also, we do not need the unperturbed Hamiltonian to have positive-definite normal torsion.

This paper completes the investigations undertaken in [25, 26] where some non-generic conditions, or some conditions difficult to verify in concrete systems, were assumed. The present paper assumes very general conditions and provides a new topological mechanism of diffusion based on Aubry-Mather sets.

2. Main result

In this section we state the assumptions and the main result of this paper. After the statement, the assumptions and the main result are explained and exemplified.

(A1) M is a n-dimensional C^r-differentiable Riemannian manifold, and $f : M \to M$ is a C^r-smooth map, for some $r \geq 2$.

(A2) There exists a 2-dimensional submanifold Λ in M, diffeormorphic to an annulus $\Lambda \simeq \mathbb{T}^1 \times [0, 1]$. We assume that f is 2-normally hyperbolic to Λ in M (see Subsection 5.1 for the definition). Since f is 2-normally hyperbolic to Λ, $W^s(\Lambda)$, $W^u(\Lambda)$, and Λ are all C^2-differentiable. Denote the dimensions of the stable and unstable manifolds of a point $x \in \Lambda$ by $\dim(W^s(x)) = n_s$ and $\dim(W^u(x)) = n_u$. Then, $n = 2 + n_s + n_u$.

(A3) On Λ there is a system of angle-action coordinates (ϕ, I), with $\phi \in \mathbb{T}^1$ and $I \in [0, 1]$. The restriction $f|_\Lambda$ of f to Λ is a boundary component preserving, area preserving, monotone twist map, with respect to the angle-action coordinates (ϕ, I).

(A4) The stable and unstable manifolds of Λ, $W^s(\Lambda)$ and $W^u(\Lambda)$, have a differentiably transverse intersection along a 2-dimensional homoclinic channel Γ. Since the manifolds $W^s(\Lambda)$ and $W^u(\Lambda)$ are C^2 and transverse, Γ is C^2. We assume that the scattering map S associated to Γ is well defined, and hence is C^1. See Subsection 5.1.

(A5) There exists a bi-infinite sequence of Lipschitz primary invariant tori $\{T_i\}_{i \in \mathbb{Z}}$ in Λ, and a bi-infinite, increasing sequence of integers $\{i_k\}_{k \in \mathbb{Z}}$ with the following properties:

  (i) Each torus T_i intersects the domain U^- and the range U^+ of the scattering map S associated to Γ.

  (ii) For each $i \in \{i_k + 1, \ldots, i_{k+1} - 1\}$, the image of $T_i \cap U^-$ under the scattering map S is topologically transverse to T_{i+1}.

  (iii) For each torus T_i with $i \in \{i_k + 2, \ldots, i_{k+1} - 1\}$, the restriction of f to T_i is topologically transitive.

  (iv) Each torus T_i with $i \in \{i_k + 2, \ldots, i_{k+1} - 1\}$, can be C^0-approximated from both sides by other primary invariant tori from Λ.

We will refer to a finite sequence $\{T_i\}_{i=i_k+1,\ldots,i_{k+1}}$ as above as a transition chain of tori.

(A6) The region in Λ between T_{i_k} and $T_{i_{k+1}}$ contains no invariant primary torus in its interior.

(A7) Inside each region between T_{i_k} and T_{i_k+1} there is prescribed a finite collection of Aubry-Mather sets $\{\Sigma_{\omega_1^k}, \Sigma_{\omega_2^k}, \ldots, \Sigma_{\omega_{s_k}^k}\}$, where $s_k \geq 1$, and ω_s^k denotes the rotation number of $\Sigma_{\omega_s^k}$. Each Aubry-Mather set $\Sigma_{\omega_s^k}$ is assumed to lie on some essential circle $C_{\omega_s^k}$, with the circles $C_{\omega_s^k}$ mutually disjoint for all $s \in \{1, \ldots, s_k\}$, and $C_{\omega_s^k}$ below $C_{\omega_{s'}^k}$ for all $\omega_s^k < \omega_{s'}^k$. The vertical ordering of these circles is relative to the I-coordinate on the annulus.

Instead of (A6) we can consider the following condition:

(A6′) The region Λ_k in Λ between T_{i_k} and T_{i_k+1} contains finitely many invariant primary tori $\{\Upsilon_{h_1^k}, \ldots, \Upsilon_{h_{l_k}^k}\}$, where $l_k \geq 1$, satisfying the following properties:

  (i) Each $\Upsilon_{h_j^k}$ falls in one of the following two cases:

    (a) $\Upsilon_{h_j^k}$ is an isolated invariant primary torus, i.e., an invariant torus that has a neighborhood in the annulus that does not contain any other invariant primary torus inside.

    (b) There exists a hyperbolic periodic orbit in Λ such that its stable and unstable manifolds coincide. Each invariant manifold has two branches, an upper branch and a lower branch (where the vertical ordering is relative to the I-coordinate of the annulus). Then $\Upsilon_{h_j^k}$ is an invariant primary torus consisting of the hyperbolic periodic orbit together with the upper branches of the invariant manifolds, or consisting of the hyperbolic periodic orbit together with the lower branches of the invariant manifolds.

  (ii) The invariant primary tori $\{\Upsilon_{h_1^k}, \ldots, \Upsilon_{h_{l_k}^k}\}$ are vertically ordered, in the sense that $\Upsilon_{h_j^k}$ is below $\Upsilon_{h_{j+1}^k}$, for all $j = 1, \ldots, l_k - 1$. The vertical ordering of these tori is relative to the I-coordinate on the annulus.

  (iii) For each $\Upsilon_{h_j^k}$, $j = 1, \ldots, l_k$, the inverse image $S^{-1}(\Upsilon_{h_j^k} \cap U^+)$ forms with $\Upsilon_{h_j^k}$ a topological disk $D_{h_j^k} \subseteq U^-$ below $\Upsilon_{h_j^k}$, such that $S(D_{h_j^k}) \subseteq U^+$ is a topological disk above $\Upsilon_{h_j^k}$, which is bounded by $\Upsilon_{h_j^k}$ and $S(\Upsilon_{h_j^k} \cap U^-)$. See Fig. 1.

Now we state the main result of the paper.

Theorem 2.1. *Let $f : M \to M$ be a C^r-differentiable map, and let $(T_i)_{i \in \mathbb{Z}}$ be a sequence of invariant primary tori in Λ, satisfying the properties (A1) – (A6), or (A1)-(A5) and (A6′), from above. Then for each sequence $(\epsilon_i)_{i \in \mathbb{Z}}$ of positive real numbers, there exist a point $z \in M$ and a bi-infinite increasing sequence of integers $(N_i)_{i \in \mathbb{Z}}$ such that*

$$(2.1) \qquad d(f^{N_i}(z), T_i) < \epsilon_i, \text{ for all } i \in \mathbb{Z}.$$

In addition, if condition (A7) is assumed, and some positive integers $\{n_s^k\}_{s=1,\ldots,s_k}$, $k \in \mathbb{Z}$ are given, then there exist $z \in M$ and $(N_i)_{i \in \mathbb{Z}}$ as in (2.1), and positive integers $\{m_s^k\}_{s=1,\ldots,s_k}$, $k \in \mathbb{Z}$, such that, for each k and each $s \in \{1, \ldots, s_k\}$, we have

$$(2.2) \qquad \pi_\phi(f^j(w_s^k)) < \pi_\phi(f^j(z)) < \pi_\phi(f^j(\bar{w}_s^k)),$$
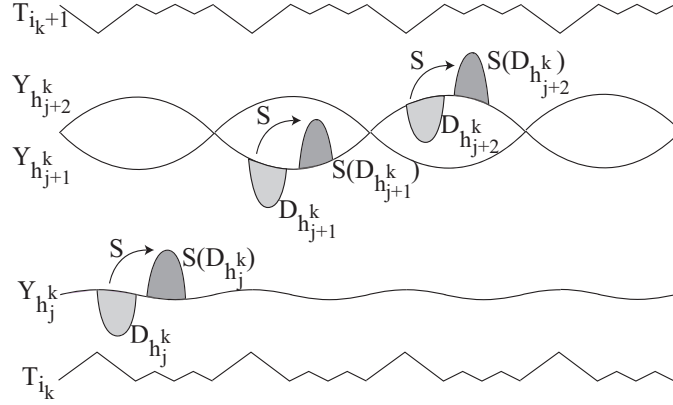
FIGURE 1. An illustration of the condition (A6′). In the figure, inside the region between T_{i_k} and T_{i_k+1}, three invariant primary tori are shown, $\Upsilon_{h_j^k}$, $\Upsilon_{h_{j+1}^k}$, $\Upsilon_{h_{j+2}^k}$, where $\Upsilon_{h_j^k}$ is as in (A6′-i-a), and $\Upsilon_{h_{j+1}^k}$, $\Upsilon_{h_{j+2}^k}$ are as in (A6′-i-b). The vertical ordering of the tori stated condition (A6′-ii) is depicted. For each of the tori $\Upsilon_{h_j^k}$, $\Upsilon_{h_{j+1}^k}$, $\Upsilon_{h_{j+2}^k}$, the condition (A6′-iii) is also illustrated.

for some $w_s^k, \bar{w}_s^k \in \Sigma_{\omega_s^k}$ and for all j with

$$N_{i_k} + \sum_{t=0}^{s-1} n_t^k + \sum_{t=0}^{s-1} m_t^k \le j \le N_{i_k} + \sum_{t=0}^{s} n_t^k + \sum_{t=0}^{s-1} m_t^k.$$

Theorem 2.1 asserts that if the conditions (A1)-(A6), or (A1)-(A5) and (A6′) are satisfied, then there exists an orbit that shadows all tori in the transition chains in the prescribed order, and also crosses over the large gaps that separate the successive transition chains. In particular, there exists an orbit that travels arbitrarily far with respect to the action variable, and there also exists an orbit that executes chaotic excursions. Additionally, if some Aubry-Mather sets are prescribed inside each gap that separates the successive transition chains, as in condition (A7), then there exists an orbit that, besides shadowing the transition chains, it also shadows the Aubry-Mather sets in the prescribed order. Note that the tori in the transition chains are shadowed in the sense that the diffusing orbit gets arbitrarily close to these tori. However, the Aubry-Mather sets are shadowed in the sense of the cyclical ordering: for each prescribed Aubry-Mather set, the diffusing orbit stays between the orbits of two points in the Aubry-Mather set, relative to the ϕ-coordinate, for any prescribed time interval.

Now we explain each assumption.

Assumption (A1) describes a C^r-differentiable, discrete dynamical system. In applications, the map f represents the first return map to a Poincaré section associated to a flow. In many examples of interest the flow is a Hamiltonian flow.

Assumption (A2) prescribes the existence of a normally hyperbolic invariant manifold Λ for f, which is diffeomorphic to an annulus. The differentiability class r and the contraction and expansion rates along the stable and unstable bundles on Λ are chosen so that the manifolds Λ, $W^u(\Lambda)$, $W^s(\Lambda)$, Γ are at least C^2-differentiable,

and the scattering map S associated to Γ is at least C^1-differentiable. The relations between the rates and the differentiability of these objects is given explicitly in Subsection 5.1.

In many examples of nearly integrable Hamiltonian systems, e.g. [2], one can identify a normally hyperbolic invariant manifold Λ_0 in the unperturbed system, and use the standard theory of normal hyperbolicity to establish the persistence of a normally hyperbolic invariant manifold Λ_ε diffeomorphic to Λ_0 for the perturbed system, for all sufficiently small perturbation parameters $\varepsilon \neq 0$. There also exist examples, e.g. from celestial mechanics, were the existence of a normally hyperbolic manifold can be established through a computer assisted proof (see [9]). We note that the assumption (A2) does not require that the stable and unstable manifolds of Λ have equal dimensions, thus our setting includes dynamical systems that are not Hamiltonian.

Assumption (A3) is satisfied automatically in examples like the weakly-coupled pendulum-rotator system considered in [18], or the periodically perturbed geodesic flow on a torus considered in [17, 33]. Some properties of area preserving, monotone twist maps of the annulus are reviewed in Section 4.

Assumption (A4) asserts that the stable and unstable manifolds of Λ have a transverse intersection Γ along a homoclinic manifold Γ, such that the scattering map associated to Γ is well defined, and hence is C^1. See Subsection 5.1. In perturbed systems one often uses a Melnikov method to establish the existence, and the persistence for all sufficiently small values of the perturbation, of a transverse intersection of the invariant manifolds.

Assumption (A5) prescribes the existence of a bi-infinite collection $\{T_i\}_{i \in \mathbb{Z}}$ of invariant primary Lipschitz tori that can be grouped into transition chains of the type $\{T_i\}_{i=i_k+1,\ldots,i_{k+1}}$.

Assumption (A5)-(i) requires that each torus T_i intersects the domain and the range of the scattering map.

Assumption (A5)-(ii) requires that each torus in a transition chain $\{T_i\}_{i=i_k+1,\ldots,i_{k+1}}$ is mapped by the scattering map topologically transversally across the next torus in the chain. Since the tori are only Lipschitz, topological transversality (topological crossing) is used in place of differentiable transversality. The definition of topological crossing can be found in [8]. Roughly speaking, two manifolds are topologically crossing if they can be made differentiably transverse with non-zero oriented intersection number by the means of a sufficiently small homotopy. In our case the two manifolds are two 1-dimensional arcs of $S(T_i)$ and of T_{i+1} in Λ. From (A5)-(ii), it follows as in [19] that $W^u(T_i)$ has a topologically transverse intersection point with $W^s(T_{i+1})$. Condition (A5)-(iii) requires that each torus in $\{T_i\}_{i=i_k+1,\ldots,i_{k+1}}$, except for the end tori, are topologically transitive. Assumption (A5)-(iv) says that all tori in the transition chain except for the end ones can be C^0-approximated from both ways by some other invariant primary tori in Λ, not necessarily from the transition chain. This means that for each $i \in \{i_k+2,\ldots,i_{k+1}-1\}$ there exist two sequences of invariant primary tori $(T_{j_l^-(i)})_{l \geq 1}$, $(T_{j_l^+(i)})_{l \geq 1}$ in Λ that approach T_i in the C^0-topology, such that the annulus bounded by $T_{j_l^-(i)}$ and $T_{j_l^+(i)}$ contains T_i in its interior for all l.

Assumption (A6) says that every pair of successive transition chains $\{T_i\}_{i=i_{k-1}+1,\ldots,i_k}$ and $\{T_i\}_{i=i_k+1,\ldots,i_{k+1}}$ is separated by the region between T_{i_k} and T_{i_k+1} which contains no invariant primary torus in its interior. A region in an

annulus that is bounded by two invariant primary tori and contains no invariant primary torus in its interior is referred as a Birkhoff Zone of Instability (BZI). The boundary tori have in general only Lipschitz regularity.

Assumptions (A5) and (A6) describe a geometric structure that is typical for the large gap problem for *a priori* unstable Hamiltonian systems. In such systems, Melnikov theory implies that $W^u(\Lambda)$ intersects transversally $W^s(\Lambda)$ at an angle of order ε, where ε is the size of the perturbation. The KAM theorem yields a Cantor family of smooth, invariant primary tori that survives the perturbation. The family of tori is interrupted by 'large gaps' of order $\varepsilon^{1/2}$ located at the resonant regions. Using the transverse intersection between $W^u(\Lambda)$ and $W^s(\Lambda)$, one can find heteroclinic connections between KAM tori that are sufficiently close, within order ε, from one another, and thus form transition chains of tori. Since the large gaps are of order $\varepsilon^{1/2}$ and the splitting size of $W^u(\Lambda)$, $W^s(\Lambda)$ is only order ε, the transition chain mechanism cannot be extended across the large gaps. In our model, the large gaps are modeled by BZI's, as in assumption (A6). This can be achieved by extending the transition chains to maximal transition chains, that go from the boundary of one large gap to the boundary of the next large gap. The intermediate tori in the chain can be chosen as KAM tori: therefore the assumption that these tori are topologically transitive and are C^0-approximable from both sides by other tori is satisfied in such cases. This may not be the case for the tori at the ends of the transition chains.

Assumption (A7) says that inside each BZI between T_{i_k} and T_{i_k+1} there is a prescribed collection of Aubry-Mather sets $\{\Sigma_{\omega_1^k}, \Sigma_{\omega_2^k}, \ldots, \Sigma_{\omega_{s_k}^k}\}$ that is vertically ordered. The vertical ordering means that the Aubry-Mather sets lie on essential (non-invariant) circles $C_{\omega_s^k}$ that are graphs over the ϕ-coordinate of the annulus, and with $C_{\omega_s^k}$ below $C_{\omega_{s'}^k}$ provided $\omega_s^k < \omega_{s'}^k$; we write $C_{\omega_s^k} \prec C_{\omega_{s'}^k}$. The vertical ordering of the Aubry-Mather sets is shown for example in [27].

Assumption (A6$'$) is a relaxation of (A6). Instead of requiring that the region in Λ between T_{i_k} and T_{i_k+1} is a BZI, it allows the existence of finitely many invariant primary tori $\{\Upsilon_{h_j^k}\}_{j=1,\ldots,l_k}$ that separate the region into disjoint components. These invariant primary tori are either isolated or else they consist of hyperbolic periodic points together with branches of their stable and unstable manifolds which are assumed to coincide. We require that the image of each $\Upsilon_{h_j^k}$ under S satisfies a certain transversality condition with $\Upsilon_{h_{j+1}^k}$ that allows one to use the scattering map in order to move points from one side of the set to the other side of the set. We note that isolated invariant tori and hyperbolic periodic points whose stable and unstable manifolds coincide do not occur in generic systems.

3. Application

We apply Theorem 2.1 to show the existence of diffusing orbits in an example of a nearly integrable Hamiltonian system. Let

$$(3.1) \qquad H_\varepsilon(p,q,I,\phi,t) \quad = \quad h_0(I) \pm \left(\frac{1}{2}p^2 + V(q)\right) + \varepsilon h(p,q,I,\phi,t;\varepsilon),$$

where $(p,q,I,\phi,t) \in \mathbb{R} \times \mathbb{T}^1 \times \mathbb{R} \times \mathbb{T}^1 \times \mathbb{T}^1$ and h is a trigonometric polynomial in (ϕ,t). Here $h_0(I)$ represents a rotator, $P_\pm(p,q) = \pm(\frac{1}{2}p^2 + V(q))$ represents a pendulum, and εh a small, periodic coupling. We assume that V, h_0 and h are uniformly C^r for some r sufficiently large. We assume that V is periodic in q of

period 1 and has a unique non-degenerate global maximum; this implies that the pendulum has a homoclinic orbit $(p^0(\sigma), q^0(\sigma))$ to $(0,0)$, with $\sigma \in \mathbb{R}$. We also assume that h_0 satisfies a uniform twist condition $\partial^2 h_0/\partial I^2 > \theta$, for some $\theta > 0$, and for all I in some interval (I^-, I^+), with $I^- < I^+$ independent of ε.

The Melnikov potential for the homoclinic orbit $(p^0(\sigma), q^0(\sigma))$ is defined by

$$\mathcal{L}(I, \phi, t) = \quad - \int_{-\infty}^{\infty} \ \left[h(p^0(\sigma), q^0(\sigma), I, \phi + \omega(I)\sigma, t + \sigma; 0)\right.$$
$$\left. -h(0, 0, I, \phi + \omega(I)\sigma, t + \sigma; 0)\right] d\sigma,$$

where $\omega(I) = (\partial h_0/\partial I)(I)$.

We assume the following non-degeneracy conditions on the Melnikov potential:

(i) For each $I \in (I^-, I^+)$, and each (ϕ, t) in some open set in $\mathbb{T}^1 \times \mathbb{T}^1$, the map

$$\tau \in \mathbb{R} \to \mathcal{L}(I, \phi - \omega(I)\tau, t - \tau) \in \mathbb{R}$$

has a non-degenerate critical point τ^*, which can be parameterized as

$$\tau^* = \tau^*(I, \phi, t).$$

(ii) For each (I, ϕ, t) as above, the function

$$(I, \phi, t) \to \frac{\partial \mathcal{L}}{\partial \phi}(I, \phi - \omega(I)\tau^*, t - \tau^*)$$

is non-constant, negative in the case of P_-, and positive in the case of P_+.

This example and the above conditions are considered in [18]. There are some additional non-degeneracy conditions on h and $\partial h/\partial \varepsilon$ that are required in [18]; we do not need to assume those conditions here.

Now we verify the conditions (A1)-(A6) from Section 2 for this model. We will rely heavily on the estimates from [18].

Condition (A1). The time-dependent Hamiltonian in (3.1) is transformed into an autonomous Hamiltonian by introducing a new variable A, symplectically conjugate with t obtaining

$$(3.2) \qquad \tilde{H}_\varepsilon(p, q, I, \phi, A, t) = h_0(I) \pm (\tfrac{1}{2}p^2 + V(q)) + A + \varepsilon h(p, q, I, \phi, t; \varepsilon),$$

where $(p, q, I, \phi, A, t) \in (\mathbb{R} \times \mathbb{T}^1)^3$. We fix an energy manifold $\{\tilde{H}_\varepsilon = \tilde{h}\}$ for some $\tilde{h}$, and restrict to the Poincaré section $\{t = 1\}$ for the Hamiltonian flow. The resulting manifold is a 4-dimensional manifold M_ϵ parametrized by some coordinates $(p_\varepsilon, q_\varepsilon, I_\varepsilon, \phi_\varepsilon)$. The first return map to M_ε of the Hamiltonian flow is a C^r-differentiable map f_ϵ.

Condition (A2). In the unperturbed case $\varepsilon = 0$, the manifold

$$\Lambda_0 := \{(p, q, I, \phi) \,|\, p = q = 0\}$$

is a normally hyperbolic invariant manifold for f_0. The dynamics on Λ_0 is given by an integrable twist map, and Λ_0 is foliated by invariant 1-dimensional tori. For the perturbed system, Λ_0 can be continued to a manifold Λ_ε diffeomorphic to Λ_0, that is locally invariant for the perturbed flow for all ε sufficiently small. The theory of normal hyperbolicity [30] can be used as in [18] to show that Λ_ε can be extended so that it is a normally hyperbolic invariant manifold for the flow. Each point in Λ_ε has 1-dimensional stable and unstable manifolds. The regularity of h_0 and the uniform twist condition allows one to apply the KAM theorem and conclude

the existence of a KAM family of primary invariant tori in Λ_ε that survives the perturbation, for all ε sufficiently small.

Condition (A3). The map f_ε is symplectic. This plus the twist condition on h_0 implies that f_ε restricted to Λ_ε is an area preserving, monotone twist map.

Condition (A4). The non-degeneracy conditions on the Melnikov function imply that $W^u(\Lambda_\varepsilon)$ and $W^s(\Lambda_\varepsilon)$ have a transverse intersection along a homoclinic manifold Γ_ε, provided ε is sufficiently small. Moreover, it is shown in [18, 19] that by restricting to some convenient homoclinic manifold Γ_ε one can ensure that the maps $\Omega_\varepsilon^\pm : \Gamma_\varepsilon \to \Lambda_\varepsilon$ are diffeomorphisms onto their images; thus the scattering map $S_\varepsilon : U_\varepsilon^- \to U_\varepsilon^+$ is a diffeomorphism between some two open sets $U_\varepsilon^-, U_\varepsilon^+ \subseteq \Lambda_\varepsilon$, of size $O(1)$. For ε fixed to some sufficiently small value, we let $\Gamma := \Gamma_\varepsilon$ and $S := S_\varepsilon$.

Conditions (A5),(A6) and (A6′) The paper [18] applies an averaging procedure to reduce the dynamics on Λ_ε to a normal form up to $O(\varepsilon^2)$ away from resonances. The averaging procedure fails within the resonant regions, corresponding to the values $I_\varepsilon(k,l)$ of the action variable where $k\omega(I) + l = 0$. A resonance is said to be of order j if the j-th order averaging cannot be applied about the corresponding action level set.

Since h is a trigonometric polynomial, one has to deal with only finitely many resonant regions. Outside the resonant regions one applies the KAM theorem and obtain KAM tori that are at a distance of order $O(\varepsilon^{3/2})$ from one another. The resonant regions yield gaps between KAM tori of size $O(\varepsilon^{j/2})$, where j is the order of the resonance. Only the resonances of order 1 and 2 are of interest, as they produce gaps of size $O(\varepsilon)$ and $O(\varepsilon^{1/2})$ respectively. Inside each resonant region, the system can be approximated by a system similar to a pendulum. In such a region, under appropriate non-degeneracy conditions, it is shown that there exist primary KAM tori close to the separatrices of the pendulum, secondary KAM tori (homotopically trivial), and stable and unstable manifolds of hyperbolic periodic orbits that pass close to the separatrices of the pendulum. Moreover, these objects can be chosen to be $O(\varepsilon^{3/2})$ from one another. In the generic case when the stable and unstable manifolds of hyperbolic periodic orbit intersect transversally, and there are no isolated invariant tori, a resonant region determines a BZI as in (A6). In the non-generic case when the stable and unstable manifolds of a hyperbolic periodic orbit coincide, or there exist isolated invariant tori, the resonant region is as described as in (A6′). The estimates from [18] imply that there exist primary KAM tori that are within $O(\varepsilon^{3/2})$ from the boundaries of the gap, or to the stable and unstable manifolds of the hyperbolic periodic orbits inside the resonant regions. These estimates do not allow one to precisely locate the boundaries of the BZI's or to say anything about their dynamics.

The Melnikov conditions imply that the scattering map S_ε associated to this homoclinic channel Γ_ε can be computed in terms of the Melnikov potential $\mathcal{L}$. If $S_\varepsilon(x^-) = x^+$, then the change in the I_ε-coordinate under S_ε is given by

$$(3.3) \qquad I_\varepsilon(x^+) - I_\varepsilon(x^-) = -\varepsilon \frac{\partial \mathcal{L}}{\partial \phi}(I_\varepsilon, \phi_\varepsilon - \omega(I_\varepsilon)\tau^*, t - \tau^*) + O_{C^1}(\varepsilon^{1+\varrho}),$$

for some $\varrho > 0$. Condition (ii) implies that there are points in the domain of the scattering map S whose I_ε-coordinate is increased by $O(\varepsilon)$ under S.

We can use these estimates to construct transition chains of invariant primary tori alternating with gaps, as in (A5) and (A6), or as in (A5) and (A6′). For ε

sufficiently small and fixed, we let $M := M_\varepsilon$, $f := f_\varepsilon$, and Λ be the annulus in Λ_ε bounded by a pair of tori T_{I_a}, T_{I_b} with $I^- < I_a < I_b < I^+$.

First, we choose a sequence of resonant regions and non-resonant regions that intersect the domain U^- and the range U^+ of the scattering map. Since the KAM primary tori are within $O(\varepsilon^{3/2})$ from one another, and the scattering map makes jumps of order $O(\varepsilon)$ in the increasing direction of I_ε, then we can find smooth KAM primary tori $\{T_{i_k+2}, T_{i_k+2}, \ldots, T_{i_{k+1}-1}\}$ such that $W^u(T_i)$ has a transverse intersection with $W^s(T_{i+1})$ for all $i \in \{i_k + 2, i_k + 3, \ldots, i_{k+1} - 1\}$, and that T_{i_k+2} and $T_{i_{k+1}-1}$ are within $O(\varepsilon^{3/2})$ from the separatrices of the penduli corresponding to two consecutive resonant gaps of orderer 1 or 2. The dynamics on each such a torus is quasi-periodic, so is topologically transitive. This ensures condition (A5)-(iii). Moreover, we can choose these KAM tori so that they are 'interior' to the Cantor family of tori, i.e. they can be approximated from both sides by other KAM primary tori. This ensures condition (A5)-(iv). To the transition chain $\{T_{i_k+2}, T_{i_k+3}, \ldots, T_{i_{k+1}-1}\}$ we add, at each end, a torus T_{i_k+1} and a torus $T_{i_{k+1}}$. These end tori bound resonant gaps that are either BZI's or consist of hyperbolic periodic orbits together with their invariant manifolds. Since $T_{i_k+1}, T_{i_{k+1}}$ are within $O(\varepsilon^{3/2})$ from $T_{i_k+2}, T_{i_{k+1}-1}$, respectively, and the scattering map S_ε makes jumps by order $O(\varepsilon)$, it follows that $S(T_{i_k+1})$ topologically crosses T_{i_k+2}, and $S(T_{i_{k+1}-1})$ topologically crosses $T_{i_{k+1}}$. This ensures condition (A5)-(ii). Condition (A1)-(i) is ensured automatically by our initial choice of the resonant regions and the non-resonant regions so that they intersect the domain U^- and the range U^+ of S_ε. The end tori T_{i_k+1} and $T_{i_{k+1}}$ are at the boundaries of two consecutive resonant gaps. This construction is continued for all resonant and non-resonant regions. Thus, for ε fixed and sufficiently small, we obtain sequences of tori $\{T_{i_k+1}, T_{i_k+2}, \ldots, T_{i_{k+1}}\}$ as in (A5), interspersed with gaps between T_{i_k} and T_{i_k+1}, and also between $T_{i_{k+1}}$ and $T_{i_{k+1}+1}$, as in (A6).

We are under the assumption of Theorem 2.1. Then there exists a diffusing orbit that shadows the transition chains of invariant primary tori and crosses the prescribed gaps. In particular, if we choose an initial torus T_{I_a} and a final torus T_{I_b} so that they are $O(1)$ apart, we obtain a diffusing orbit whose action variable changes by $O(1)$. We note our theorem applies even for the choice of the pendulum P_-, when the unperturbed Hamiltonian does not have positive-definite normal torsion. The assumption of positive definiteness seems to be very important for variational methods.

We emphasize that, although we are using many of the estimates from [18], we obtain a different mechanism of diffusion. Our mechanism still involves transition chains of invariant primary tori, but uses the inner dynamics restricted to the normally hyperbolic invariant manifold to cross over the large gaps. The paper [18] identifies secondary tori and hyperbolic invariant manifolds of lower dimensional tori inside the large gaps, and forms transition chains of such objects that can be joined with the transition chains of primary tori. Hence, it still uses the outer dynamics to cross over those gaps.

Since in our approach we do not use transition chains of secondary tori or of hyperbolic invariant manifolds of lower dimensional tori, we do not need to assume the additional non-degeneracy conditions on the scattering map acting on these objects as in [18].

Moreover, one can combine the topological mechanism in this paper with the one in [23] and obtain diffusing orbits that visit any given collection of primary tori, secondary tori, invariant manifolds of lower dimensional tori, and Aubry-Mather sets, in any prescribed order.

4. Background on twist maps and Aubry-Mather sets

Let $\tilde{A} = \mathbb{T}^1 \times [0,1] = \{(x,y) \in \mathbb{T}^1 \times [0,1]\}$ be an annulus, and let $A = \mathbb{R} \times [0,1]$ be its universal cover with the natural projection $\pi : A \to \tilde{A}$ given by $\pi(x,y) = (\tilde{x}, \tilde{y})$, where $\tilde{x} = x(\mathrm{mod}\, 1)$ and $\tilde{y} = y$. Let π_x be the projection onto the first component, and π_y be the projection onto the second component. Let $\tilde{f} : \tilde{A} \to \tilde{A}$ be a C^1 mapping on $\tilde{A}$, and let $f : A \to A$ be the unique lift of $\tilde{f}$ to A satisfying $\pi_x(f(0,0)) \in [0,1)$ and $\pi \circ f = \tilde{f} \circ \pi$. In order to simplify the notation, below we will not make distinction between $\tilde{A}$ and A, and between $\tilde{f}$ and f.

We assume that f is orientation preserving, boundary preserving, area preserving, and its satisfies a monotone twist condition, i.e., $|\partial(\pi_x \circ f)/\partial y| > 0$ at all points in the annulus.

We note that the above properties imply that f is exact symplectic, i.e. f has zero flux, meaning that for any rotational curve γ the area of the regions above γ and below $f(\gamma)$ equals the area below γ and above $f(\gamma)$.

In the sequel we will assume that f is a positive twist, meaning that $\partial(\pi_x \circ f)/\partial y > 0$ at all points. The map f restricted to the boundary components $\mathbb{T}^1 \times \{0\}$, $\mathbb{T}^1 \times \{1\}$ of the annulus has well defined rotation numbers ω_-, ω_+, respectively, with $\omega^- < \omega^+$. We will assume that $\omega^-, \omega^+ > 0$.

By an invariant primary torus (essential invariant circle) we mean a 1-dimensional torus T invariant under f that cannot be homotopically deformed into a point inside the annulus. Since f is a monotone twist map, each invariant primary torus T is the graph of some Lipschitz function (see [4, 5]).

A region in A between two invariant primary tori T_1 and T_2 is called a Birkhoff Zone of Instability (BZI) provided that there is no invariant primary torus in the interior of the region.

It is known that, for an area preserving monotone twist map f of A, given a BZI, there exist Birkhoff connecting orbits that go from any neighborhood of one boundary torus to any neighborhood of the other boundary torus (see [4, 5, 37]). We have the following results:

Theorem 4.1 (Birkhoff Connecting Theorem). *Suppose that T_1 and T_2 bound a BZI. For every pair of neighborhoods U of T_1 and V of T_2 there exist a point $z \in U$ and an integer $N > 0$ such that $f^N(z) \in V$.*

Corollary 4.2. *Suppose that T_1 and T_2 bound a BZI, and that the restrictions of f to T_1 and T_2 are topologically transitive. For every $\zeta_1 \in T_1, \zeta_2 \in T_2$ and every pair of neighborhoods U of ζ_1 and V of ζ_2, there exist a point $z \in U$ and an integer $N > 0$ such that $f^N(z) \in V$.*

A subset $M \subseteq A$ is said to be monotone (cyclically ordered) if $\pi_x(z_1) < \pi_x(z_2)$ implies $\pi_x(f(z_1)) < \pi_x(f(z_2))$ for all $z_1, z_2 \in M$. For $z \in A$ the extended orbit of z is the set $EO(z) = \{f^n(z) + (j,0) : n, j \in \mathbb{Z}\}$. The orbit of z is said to be monotone (cyclically ordered) if the set $EO(z)$ is monotone. If the orbit of $z \in A$ is monotone, then the rotation number $\rho(z) = \lim_{n \to \infty} (\pi_x(f^n(z))/n)$ exists. We

denote $\mathrm{Rot}(\omega) = \{z \in A \,:\, \rho(z) = \omega\}$. All points in the same monotone set have the same rotation number.

Definition 4.3. An Aubry-Mather set for $\omega \in \mathbb{T}^1$ is a minimal, monotone, f-invariant subset of $\mathrm{Rot}(\omega)$.

Here by a minimal set we mean a closed invariant set that does not contain any proper closed invariant subsets. (Equivalently, the orbit of every point in the set is dense in the set.) This should not be confused with action-minimizing or h-minimal sets, where h is a generating function for f.

Theorem 4.4 (Aubry-Mather Theorem). *For every $\omega \in [\omega^-, \omega^+]$, there exists a non-empty Aubry-Mather set Σ_ω in $\mathrm{Rot}(\omega)$.*

Aubry-Mather sets defined as above can be obtained as limits of monotone Birkhoff periodic orbits [34]. There may be many Aubry-Mather sets with the same rotation number [45]. On the other hand, if one requires Aubry-Mather sets to be action minizing, there exists a unique recurrent Aubry-Mather set for any given irrational rotation number.

In the sequel we will use the following result on the vertical ordering of Aubry-Mather sets from [27].

Theorem 4.5. *There exists a family of essential circles C_ω in A for $\omega \in [\omega^-, \omega^+]$ such that:*

  (i) *Each C_ω is a graph over $y = 0$;*
  (ii) *The circles C_ω are mutually disjoint, and if $\omega' > \omega$ then $C_{\omega'}$ is above C_ω;*
  (iii) *Each C_ω contains an Aubry-Mather set Σ_ω.*

The above circles have Lipschitz regularity, and are projections of so called 'ghost circles' that are objects in $\mathbb{R}^{\mathbb{Z}}$. See [27] for details. A similar result to Theorem 4.5 appears in [35] who find Aubry-Mather sets lying on pseudo-graphs that are (not strictly) vertically ordered.

There are some analogues of the Birkhoff Connecting Theorem for Aubry-Mather sets. The following lemma is used in [33] to provide a topological proof for Mather Connecting Theorem stated below.

Lemma 4.6. *Suppose that T_1 and T_2 bound a BZI. Let Σ_ω be an Aubry-Mather set of rotation number ω inside the BZI. Let p be a recurrent point in Σ_ω and $W(p)$ be a neighborhood of p inside the BZI. The following hold true:*

  (i) *For some positive number n^+ (resp. n^-) depending on $W(p)$ the set $\bigcup_{j=0}^{n^+} f^j(W(p))$ (resp. $\bigcup_{j=0}^{n^-} f^{-j}(W(p))$) separates the cylinder.*
  (ii) *The set $W^{+\infty} := \bigcup_{j=0}^{\infty} f^j(W(p))$ (resp. the set $W^{-\infty} := \bigcup_{j=0}^{\infty} f^{-j}(W(p))$), is connected and open.*
  (iii) *The closure of $W^{+\infty}$ (resp. $W^{-\infty}$) contains both boundary tori T_1 and T_2.*
  (iv) *The set $W^{\infty} := \bigcup_{j=-\infty}^{\infty} f^j(W(p))$ is invariant, and both $W^{+\infty}$ and $W^{-\infty}$ are open and dense in W^{∞}.*

The following result says that there exist orbits that visit any prescribed bi-infinite sequence of Aubry-Mather sets inside a BZI (see [46, 53, 29, 35, 33]).

Theorem 4.7 (Mather Connecting Theorem). *Suppose that T_1 and T_2 bound a BZI, and $\{\Sigma_{\omega_i}\}_{i\in\mathbb{Z}}$ is a bi-infinite sequence of Aubry-Mather sets inside the BZI. Let $\varepsilon_i > 0$ for $i \in \mathbb{Z}$. Then there exist a point z inside the BZI and an increasing*

bi-infinite sequence of integers $\{j_i\}_{i\in\mathbb{Z}}$ *such that* $f^{j_i}(z)$ *is within* ε_i *from* Σ_{ω_i} *for all* $i \in \mathbb{Z}$.

The Aubry-Mather sets in Theorem 4.7 are action minimizing. The following topological version of Mather Connecting Theorem, due to Hall [29], provides shadowing orbits of Aubry-Mather sets that are not necessarily action minimizing. This approach can be implemented in rigorous computer experiments [32].

Theorem 4.8. *Suppose that* T_1 *and* T_2 *bound a BZI, and* $\{z_s\}_{s\in\mathbb{Z}}$ *is a bi-infinite sequence of monotone* (p_s/q_s)-*periodic points, with the rotation numbers* p_s/q_s *mutually distinct, inside the BZI. Given a bi-infinite sequence* $\{n_s\}_{s\in\mathbb{Z}}$ *of positive integers, then there exist a point* z *and a bi-infinite sequence* $\{m_s\}_{s\in\mathbb{Z}}$ *of positive integers such that, for each* $s \geq 0$, *there exist some points* $w_s, \bar{w}_s$ *in the extended orbit of* z_s *such that*

$$\pi_x(f^j(w_s)) < \pi_x(f^j(z)) < \pi_x(f^j(\bar{w}_s)) \text{ for}$$

(4.1)
$$\sum_{t=0}^{s-1} n_t + \sum_{t=0}^{s-1} m_t \leq j \leq \sum_{t=0}^{s} n_t + \sum_{t=0}^{s-1} m_t.$$

A similar statement holds for each $s < 0$.

In the above, n_s represents the number of iterates for which the orbit of z shadows – in the sense of the cyclical ordering – the extended orbit of z_s, and m_s represents the number of iterates it takes the orbit of z to pass from the extended orbit of z_s to the extended orbit of z_{s+1}.

They main tool used in Hall's arguments is that of a positive (negative) diagonal. Denote by $\mathcal{Z}$ the BZI bounded by the tori T_1 and T_2. Let

(4.2) $$I_z = \{w \in \mathcal{Z} \,|\, \pi_x(w) = \pi_x(z)\},$$

(4.3) $$I_z^+ = \{w \in I_z \,|\, \pi_y(w) \geq \pi_y(z)\},$$

(4.4) $$I_z^- = \{w \in I_z \,|\, \pi_y(w) \leq \pi_y(z)\},$$

(4.5) $$B_{z_0,z_1} = \{w \in \mathcal{Z} \,|\, \pi_x(z_0) < \pi_x(w) < \pi_x(z_1)\},$$

where z, z_0, z_1 are points in the annulus.

A positive diagonal D in B_{z_0,z_1} is a set $D \subseteq \mathrm{cl}(B_{z_0,z_1})$ such that

   (i) D is simply connected and the closure of its interior;
   (ii) $\partial D \cap \mathrm{cl}(B_{z_0,z_1}) \subseteq I_{z_0}^- \cup I_{z_1}^+ \cup T_1 \cup T_2$;
   (iii) $\partial D \cap I_{z_0}^- \neq \emptyset$ and $\partial D \cap I_{z_1}^+ \neq \emptyset$.

The set $\partial D \cap B_{z_0,z_1}$ has exactly two components connecting $I_{z_0}^- \cup T_1$ to $I_{z_1}^+ \cup T_2$, which are called the upper and lower edges of D, respectively. We informally say that these components 'stretch across' B_{z_0,z_1}. See Fig. 2.

A negative diagonal and its upper and lower edges are defined similarly.

An important feature of positive diagonals is the following hereditary property. Given z_0, z_0 such that $\pi_x(z_0) < \pi_x(z_1)$ and $\pi_x(f(z_0)) < \pi_x(f(z_1))$, if D is a positive diagonal in B_{z_0,z_1}, then $f(D) \cap B_{f(z_0),f(z_1)}$ has a component D' that is a positive diagonal in $B_{f(z_0),f(z_1)}$.

One way to generate a positive diagonal set is by taking a component of the intersection between $f^k(B_{w_0,w_1})$ and $B_{f^k(w_0),f^k(w_1)}$. In this case, there exists a positive diagonal in $B_{f^k(w_0),f^k(w_1)}$ whose upper edge is contained on $f^k(I_{w_0}^+)$ and lower edge is contained in $f^k(I_{w_1}^-)$. More general, one has the following important
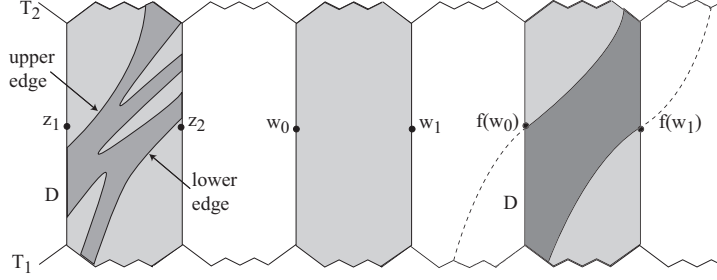
FIGURE 2. Two positive diagonal sets. The positive diagonal set
on the left has its upper and lower edges marked. The positive
diagonal set on the right is obtained by intersecting $f(B_{w_0,w_1})$
with $B_{f(w_0),f(w_1)}$. The upper edge of this diagonal set is contained
in $f(I_{w_0}^+)$ and the lower edge is contained in $f(I_{w_1}^-)$.

property. If D has the upper edge contained in $f^k(I_{w_0}^+)$ and the lower edge contained
in $f^k(I_{w_1}^-)$, and $\partial D \cap B_{z_0,z_1} \subseteq f^k(I_{w_0}^+ \cup I_{w_1}^-)$, for some w_0, w_1 with $\pi_x(w_0) < \pi_x(w_1)$
and some $k > 0$, then then $f(D) \cap B_{f(z_0),f(z_1)}$ has a component D' which can
be chosen so that its upper edge is contained in $f^{k+1}(I_{w_0}^+)$ and its lower edge is
contained in $f^{k+1}(I_{w_1}^-)$. See Fig. 2. A similar property holds for negative diagonals.

The proof of Theorem 4.8 in [29] is an inductive argument which, for a given
pair of adjacent points $w_0, \bar{w}_0$ in the extended orbit of z_0, and for each $\sigma \geq 0$, pro-
duces a nested sequence $D_0 \supseteq D_1 \supseteq \ldots \supseteq D_\sigma$ of negative diagonals of $B_{w_0,\bar{w}_0}$
such that, for each $s \in \{0, \ldots, \sigma\}$, the following hold: (a) the orbit of each
point $z \in D_s$ satisfies the ordering relation (4.1), and (b) there is a sufficiently
large $j_s > 0$ such that $f^{j_s+j}(D_s)$ contains a component that is a positive di-
agonal in $B_{f^j(w_s),f^j(\bar{w}_s)}$, for some adjacent points $w_s, \bar{w}_s \in EO(z_s)$, and for all
$j = 1, \ldots, n_s$. In the above, $j_s = \sum_{t=0}^{s-1} n_t + \sum_{t=0}^{s-1} m_t$. Moreover, in this in-
ductive argument one can choose the diagonal sets D_s so that $f^{j_s+j}(D_s)$ has
the upper edge contained in $f^{j_s+j}(I_{w_0}^+)$, lower edge contained in $f^{j_s+j}(I_{\bar{w}_0}^-)$, and
$\partial f^{j_s+j}(D_s) \cap B_{f^{j_s+j}(w_0),f^{j_s+j}(\bar{w}_0)} \subseteq f^{j_s+j}(I_{w_0}^+ \cup I_{\bar{w}_0}^-)$.

For the basis step, starting with $w_0, \bar{w}_0$ and applying the hereditary property
from above n_0 times, one obtains a negative diagonal set D_0 of $B_{w_0,\bar{w}_0}$ with the
properties that each point $z \in D_0$ satisfies the ordering relation (4.1) for $s = 0$, and
$f^{n_0}(D_0)$ has a component that is a positive diagonal of $B_{f^{n_0}(w_0),f^{n_0}(\bar{w}_0)}$.

For the inductive step, one assumes a negative diagonal D_σ of $B_{w_0,\bar{w}_0}$ as above,
and wants to produce a negative diagonal $D_{\sigma+1} \subseteq D_\sigma$ of $B_{w_0,\bar{w}_0}$ which fulfils the
corresponding properties. The key idea is to use the existence of points near $y = 0$
that get near $y = 1$, and of points near $y = 1$ that get near $y = 0$, as provided by
Theorem 4.1, in order to show that for some j_σ sufficiently large $f^{j_\sigma}(D_\sigma)$ contains a
component that stretches all the way across a fundamental interval of the annulus.
Hence $f^{j_\sigma}(D_\sigma)$ contains a subset that is a positive diagonal of $B_{w_{\sigma+1},\bar{w}_{\sigma+1}}$ for two
adjacent points $w_{\sigma+1}, \bar{w}_{\sigma+1} \in EO(z_{\sigma+1})$. From this it follows that $f^{j_\sigma+j}(D_\sigma)$
contains a component that is a positive diagonal in $B_{f^j(w_{\sigma+1}),f^j(\bar{w}_{\sigma+1})}$ for all $j = 1, \ldots, n_{\sigma+1}$. This completes the inductive step.

Applying a similar argument for the negative iterates of f produces a nested
sequence of positive diagonals of $B_{w_0,\bar{w}_0}$. A positive diagonal of $B_{w_0,\bar{w}_0}$ always
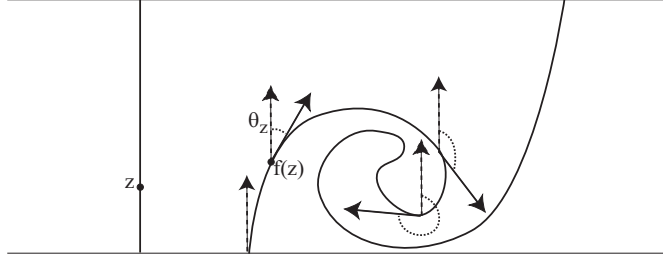
FIGURE 3. Positively tilted map.

has a non-empty intersection with a negative diagonal of $B_{w_0, \bar{w}_0}$. This implies the existence of points z whose forward orbits satisfy the ordering conditions in (4.1) and whose backwards orbits satisfy similar ordering conditions.

Using limit arguments as in [34], one can obtain shadowing of Aubry-Mather sets of irrational rotation numbers as well. These topological ideas will be used in the proof of Theorem 6.2 below.

Remark 4.9. An immediate consequence of Lemma 4.6 is that, given a neighborhood W of a point $p \in \Sigma_\omega$, where Σ_ω is an Aubry-Mather set inside a BZI bounded by T_1 and T_2, and given a neighborhood U of T_1 or T_2, there exists an arbitrarily large $j > 0$ such that $f^j(W(p)) \cap U \neq \emptyset$. Also, there exists an arbitrarily large $j' > 0$ such that $f^{-j'}(W(p)) \cap U \neq \emptyset$.

Remark 4.10. The results in this section hold if we replace conditions (i) and (ii) from the definition of an area preserving, monotone twist map with the following weaker conditions:

(i') f satisfies the following 'condition B': for every pair of neighborhoods U_1 of T_1 and U_2 of T_2, there exist $z_1, z_2 \in A$ and $n_1, n_2 > 0$ such that $z_1 \in U_1$ and $f^{n_1}(z_1) \in U_2$, and $z_2 \in U_2$ and $f^{n_2}(z_2) \in U_1$. Note that this condition only makes sense if we restrict the dynamics to a BZI.

(ii') f satisfies the following positive tilt condition: if we denote by θ_z the angle deviation from the vertical, measured from the vertical vector $(0, 1)$ to $Df_z(0, 1)$, with the clockwise direction taken as the positive direction, and defined in such a way that $\theta_{(x,0)} \in [-\pi/2, \pi/2]$ and θ is continuous, then $\theta_z > 0$ at all points. See Fig. 3.

Compositions of positive twist maps, are for example, positive tilt maps. We shall note that the Aubry-Mather theory applies to positive tilt maps as well (see [31]).

Remark 4.11. Aubry-Mather theory and the above shadowing result also hold for generalized twist maps of the higher dimensional annulus $S^1 \times \mathbb{R}^n$. See [1].

Remark 4.12. The topological approach in this section does not yield trajectories that get close to each set in the prescribed collection of Aubry-Mather sets, as in Theorem 4.7. It seems possible, however, that these topological methods can be combined with the variational methods in [46] to obtain trajectories that go very close to each Aubry-Mather set in the given collection.

5. Background on the scattering map and on the topological method of correctly aligned windows

5.1. **Scattering map.** The scattering map acts on the normally hyperbolic invariant manifold Λ and relates the past asymptotic trajectory of each orbit in the homoclinic manifold to its future asymptotic behavior. We review its properties following [19].

For the general case, we consider a C^r-differentiable manifold M, a C^r-differentiable map $f : M \to M$, and an l-dimensional normally hyperbolic invariant manifold Λ for f. By the definition of normal hyperbolicity, there exists a splitting of the tangent bundle of TM into sub-bundles

$$TM = E^u \oplus E^s \oplus T\Lambda,$$

that are invariant under df, and there exist a constant $C > 0$ and rates $0 < \lambda < \mu^{-1} < 1$, such that for all $x \in \Lambda$ we have

$$v \in E^s_x \Leftrightarrow \|Df^k_x(v)\| \leq C\lambda^k \|v\| \text{ for all } k \geq 0,$$
$$v \in E^u_x \Leftrightarrow \|Df^k_x(v)\| \leq C\lambda^{-k} \|v\| \text{ for all } k \leq 0,$$
$$v \in T_x\Lambda \Leftrightarrow \|Df^k_x(v)\| \leq C\mu^{|k|} \|v\| \text{ for all } k \in \mathbb{Z}.$$

The smoothness of the invariant objects defined by the normally hyperbolic structure depends on the rates λ and μ. The map f is said to be ℓ-*normally hyperbolic along* Λ provided that $1 \leq \ell \leq r$ is an integer satisfying $\lambda\mu^\ell < 1$, i.e., $\ell < (\log \lambda^{-1})(\log \mu)^{-1}$. Then the stable and unstable manifolds $W^s(\Lambda)$ and $W^u(\Lambda)$ and the normally hyperbolic manifold Λ are all C^ℓ-differentiable. The splitting $E^s_z = T_z(W^s(x))$ depends $C^{\ell-1}$ smoothly on z in $W^s(\Lambda)$ so $\{W^s(x) \,|\, x \in \Lambda\}$ is a $C^{\ell-1}$ foliation of $W^s(\Lambda)$. (This is stated explicitly in [50] and follows from the C^r Section Theorem of [30].) Similarly, $\{W^u(x) \,|\, x \in \Lambda\}$ is a $C^{\ell-1}$ foliation of $W^u(\Lambda)$.

Since the stable (resp. unstable) manifolds of Λ are foliated by stable (resp. unstable) manifolds of points, we have that for each $x \in W^s(\Lambda)$ (resp. $x \in W^u(\Lambda)$), there exists a unique $x^+ \in \Lambda$ (resp. $x^- \in \Lambda$) such that $x \in W^s(x^+)$ (resp. $x \in W^u(x^-)$). We define the maps $\Omega^+ : W^s(\Lambda) \to \Lambda$ by $\Omega^+(x) = x^+$ and $\Omega^- : W^u(\Lambda) \to \Lambda$ by $\Omega^-(x) = x^-$. The maps Ω^+ and Ω^- are $C^{\ell-1}$-smooth since the foliations are smooth.

We now describe the scattering map. Assume that $W^u(\Lambda)$ and $W^s(\Lambda)$ have a differentiably transverse intersection along a homoclinic l-dimensional $C^{\ell-1}$-differentiable manifold Γ. This means that $\Gamma \subseteq W^u(\Lambda) \cap W^s(\Lambda)$ and, for each $x \in \Gamma$, we have

$$
\begin{aligned}
T_xM &= T_xW^u(\Lambda) + T_xW^s(\Lambda), \\
T_x\Gamma &= T_xW^u(\Lambda) \cap T_xW^s(\Lambda).
\end{aligned}
$$
(5.1)

We assume the additional condition that for each $x \in \Gamma$ we have

$$
\begin{aligned}
T_xW^s(\Lambda) &= T_xW^s(x^+) \oplus T_x(\Gamma), \\
T_xW^u(\Lambda) &= T_xW^u(x^-) \oplus T_x(\Gamma),
\end{aligned}
$$
(5.2)

where x^-, x^+ are the uniquely defined points in Λ corresponding to x.

The restrictions $\Omega^+_\Gamma, \Omega^-_\Gamma$ of Ω^+, Ω^- to Γ are local $C^{\ell-1}$-diffeomorphisms. By replacing Γ to a submanifold of it (which, with an abuse of notation, we still denote Γ) we can ensure that $\Omega^+_\Gamma : \Gamma \to U^+, \Omega^-_\Gamma : \Gamma \to U^-$ are $C^{\ell-1}$-diffeomorphisms from Γ to the open sets U^+, U^- in Λ, respectively.

Definition 5.1. A homoclinic manifold Γ satisfying (5.1) and (5.2), and for which the corresponding restrictions of the wave maps are $C^{\ell-1}$-diffeomorphisms, is referred as a homoclinic channel.

Definition 5.2. Given a homoclinic channel Γ, the scattering map associated to Γ is the $C^{\ell-1}$-diffeomorphism $S_\Gamma = \Omega_\Gamma^+ \circ (\Omega_\Gamma^-)^{-1}$ from the open subset $U^- := \Omega_\Gamma^-(\Gamma)$ in Λ to the open subset $U^+ := \Omega_\Gamma^+(\Gamma)$ in Λ.

In the sequel we will regard S_Γ as a partially defined map, so the image of a set A by S_Γ means the set $S_\Gamma(A \cap U^-)$.

In this paper, we need the following property of the scattering map.

Proposition 5.3. *Assume that T_1 and T_2 are two invariant submanifolds of complementary dimensions in Λ. Then $W^u(T_1)$ has a topologically transverse intersection with $W^s(T_2)$ inside Γ if and only if $S_\Gamma(T_1 \cap U^-)$ has a topologically transverse intersection with $T_2 \cap U^+$ in Λ.*

Proof. Let $\bar{T}_1 = (\Omega_\Gamma^-)^{-1}(T_1 \cap U^-) \subseteq \Gamma$ and $\bar{T}_2 = (\Omega_\Gamma^+)^{-1}(T_2 \cap U^+) \subseteq \Gamma$. A topologically transverse intersection of $W^u(T_1)$ with $W^s(T_2)$ in Γ occurs if and only if $\bar{T}_1$ intersects $\bar{T}_2$ topologically transversally in Γ, which is equivalent to $S_\Gamma(T_1 \cap U^-)$ intersects topologically transversally $T_2 \cap U^+$ in Λ. $\square$

For the definition of topological transversality (topological crossing) see [8]. For the main result of this paper the normally hyperbolic invariant manifold Λ is assumed to be 2-dimensional, i.e., $l = 2$, and the invariant submanifolds T_1, T_2 in Proposition 5.3 are 1-dimensional invariant tori.

5.2. **Topological method of correctly aligned windows.** We describe briefly the topological method of correctly aligned windows. We follow [55]. See also [24, 23, 41].

Definition 5.4. An (n_1, n_2)-window in an n-dimensional manifold M, where $n_1 + n_2 = n$, is a compact subset W of M together with a homeomorphism χ from some open neighborhood of $[0,1]^{n_1} \times [0,1]^{n_2}$ in $\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ to an open subset of M, such that

$$W = \chi([0,1]^{n_1} \times [0,1]^{n_2}),$$

and with a choice of an 'exit set'

$$W^{\text{exit}} = \chi\left(\partial[0,1]^{n_1} \times [0,1]^{n_2}\right)$$

and of an 'entry set'

$$W^{\text{entry}} = \chi\left([0,1]^{n_1} \times \partial[0,1]^{n_2}\right).$$

Denote by $\pi_1 : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R}^{n_1}$ the projection onto the first component, and by $\pi_2 : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R}^{n_2}$ the projection onto the second component.

Definition 5.5. Let W_1 and W_2 be (n_1, n_2)-windows, and let χ_1 and χ_2 be the corresponding local parametrizations. Let f be a continuous map on M with $f(\text{im}(\chi_1)) \subseteq \text{im}(\chi_2)$. We say that W_1 is correctly aligned with W_2 under f if the following conditions are satisfied:

    (i) $f(W_1^{\text{exit}}) \cap (W_2) = \emptyset$ and $f(W_1) \cap W_2^{\text{entry}} = \emptyset$;

(ii) There exists $y_0 \in [0,1]^{n_2}$ such that the curve $x \in [0,1]^{n_1} \mapsto \hat{f}(x, y_0)$, where $\hat{f} := \chi_2^{-1} \circ f \circ \chi_1$, has the following properties:

$$\hat{f}_{y_0}\left([0,1]^{n_1}\right) \subseteq \mathbb{R}^{n_1} \times (0,1)^{n_2},$$
$$\hat{f}_{y_0}\left(\partial[0,1]^{n_1}\right) \subseteq \left(\mathbb{R}^{n_1} \setminus [0,1]^{n_1}\right) \times (0,1)^{n_2} = \emptyset,$$
$$\deg(\pi_1 \circ \hat{f}_{y_0}, 0) = w \neq 0.$$

We call the integer $w \neq 0$ in the above definition the degree of the alignment.

The following result is a topological version of the Shadowing Lemma.

Theorem 5.6. *Let $\{W_i\}_{i \in \mathbb{Z}}$, be a collection of (n_1, n_2)-windows in M, and let f_i be a collection of continuous maps on M. If for each $i \in \mathbb{Z}$, W_i is correctly aligned with W_{i+1} under f_i, then there exists a point $p \in W_0$ such that*

$$(f_i \circ \cdots \circ f_0)(p) \in W_{i+1}, \text{ for all } i \in \mathbb{Z}.$$

Moreover, assuming that there exists $k > 0$ such that $W_i = W_{(i \bmod k)}$ and $f_i = f_{(i \bmod k)}$ for all $i \in \mathbb{Z}$, then there exists a point p as above that is periodic in the sense

$$(f_{k-1} \circ \cdots \circ f_0)(p) = p.$$

The correct alignment of windows is robust, in the sense that if two windows are correctly aligned under a map, then they remain correctly aligned under a sufficiently small C^0-perturbation of the map. Robustness makes the method of correctly aligned windows appropriate for perturbative arguments, as well as for rigorous numerical experiments.

Also, the correct alignment satisfies a natural product property. Given two windows and a map, if each window can be written as a product of window components, and if the components of the first window are correctly aligned with the corresponding components of the second window under the appropriate components of the map, then the first window is correctly aligned with the second window under the given map. For example, if we consider a pair of windows in a neighborhood of a normally hyperbolic invariant manifold, if the center components of the windows are correctly aligned and the hyperbolic components of the windows are also correctly aligned, then the windows are correctly aligned. Although the product property is quite intuitive, its rigorous statement is rather technical, so we will omit it here. The details can be found in [23].

In Sections 7 and 8, we will consider various windows lying on one of the manifolds Λ, $W^u(\Lambda)$, $W^s(\Lambda)$, or M. Without explicit mention, every such a window will be represented by the image of a rectangle through a local parametrization of the appropriate manifold. We will also consider correct alignment relations of windows lying on the same manifold. All the correct alignment relations in the arguments presented later in this paper have degree $w = 1$.

6. Existence of Birkhoff connecting orbits

In this section we state and prove an extension of the Corollary 4.2 of the Birkhoff connecting orbit theorem, and an extension of Mather's theorem on shadowing of Aubry-Mather sets, specifically of Theorem 4.8. The methodology is based on the topological approach of Hall and on the Jordan Curve Theorem. The statements below will be used in the proof of the main theorem. Throughout the section we

assume that f is an orientation preserving, boundary preserving, area preserving, monotone twist map of the annulus A, as in Section 4, and we adopt the notation conventions from that section.

In Corollary 4.2, it was assumed that the restrictions of the map to the boundary tori of the BZI are topologically transitive, and it was inferred the existence of connecting orbits from an arbitrarily small neighborhood of some prescribed point on one boundary torus to an arbitrarily small neighborhood of some prescribed point on the other boundary torus. In the statements below we prove the same result without the topological transitivity assumption.

Theorem 6.1. *Suppose that T_1 and T_2 bound a BZI $\mathcal{Z}$. Assume that $\zeta_1 \in T_1$ and $\zeta_2 \in T_2$. Fix a pair of neighborhoods U of ζ_1 and V of ζ_2. Then there exists a point $z \in U$ and an integer $N > 0$ such that $f^N(z) \in V$ for some $N > 0$ that can be chosen arbitrarily large.*

Moreover, if we assume that U, V are chosen so that $U \cap \mathcal{Z}, V \cap \mathcal{Z}$ are topological disks, then there exists a point $z' \in \partial U$ such that $f^N(z') \in \partial V$.

Proof. Consider the neighborhood U of $\zeta_1 \in T_1$. Choose two points $\zeta_1', \zeta_1'' \in T_1 \cap U$ such that ζ_1 is between ζ_1' and ζ_1'' and the portion of T_1 between ζ_1' and ζ_1'' is contained in $\text{int}(U)$. Choose a simple curve γ_0 inside U, with endpoints at ζ_1' and ζ_1''. The curve γ_0 together with the portion of T_1 between ζ_1' and ζ_1'' determines a closed topological disk $U_0 \subseteq U$, which is a one-sided compact neighborhood of ζ_1 in $\mathcal{Z}$.

Similarly, we can choose a one-sided compact neighborhood $V_0 \subseteq V$ of $\zeta_2 \in T_2$, whose boundary consists of a simple curve η_0 connecting two points $\zeta_2', \zeta_2'' \in T_2$, with ζ_2 between ζ_2' and ζ_2'', and the portion of T_2 between ζ_2' and ζ_2''.

In this way, for proving the theorem we can consider the one-sided neighborhoods U_0, V_0 instead of U, V, respectively.

Assume first that the interior of U_0 meets some Aubry-Mather set $\Sigma_{\rho_1} \subseteq \mathcal{Z}$, and that the interior of V_0 meets some Aubry-Mather set $\Sigma_{\rho_2} \subseteq \mathcal{Z}$. Since U_0 and V_0 are neighborhoods of points in the Aubry-Mather sets Σ_{ρ_1} and Σ_{ρ_2} respectively, Theorem 4.7 yields the existence of a forward orbit that goes from U_0 to V_0. Hence there exists $N > 0$ such that $f^N(U_0) \cap V_0 \neq \emptyset$. Since $f^N(U_0)$ and V_0 are open topological disks that have common points as well as non-common points (e.g., the points in T_1 and T_2, respectively), the Jordan Curve Theorem implies that $f^N(\partial U_0) \cap \partial V_0 \neq \emptyset$.

Assume now that the interiors of U_0, V_0 do not meet any Aubry-Mather set.

We choose three Aubry-Mather sets $\Sigma_{\rho_1}, \Sigma_{\rho_1'}, \Sigma_{\rho_1''}$ in $\mathcal{Z}$, lying on three essential circles $C_{\rho_1}, C_{\rho_1'}, C_{\rho_1''}$, respectively, with $\rho_1 < \rho_1' < \rho_1''$ irrational rotation numbers, and $C_{\rho_1} \prec C_{\rho_1'} \prec C_{\rho_1''}$. The existence of such vertically ordered Aubry-Mather sets follows from Theorem 4.5.

The proof of the theorem uses the following intermediate step.

Claim. There exist $j_*' > j_* > 0$ such that $\mathcal{Z} \setminus [f^{j_*}(U_0) \cup f^{j_*'}(U_0)]$ contains a component $\mathcal{U}$ which is an open topological disk that is a neighborhood of some point in Σ_{ρ_1}. Moreover, j_*, j_*' can be chosen arbitrarily large. A similar statement holds for T_2.

Proof of the claim. Let p_1 be a point in $\Sigma_{\rho_1''}$. Let $W(p_1)$ be a small neighborhood of p_1 inside the BZI, which does not intersect Σ_{ρ_1} and $\Sigma_{\rho_1'}$. By assumption, U_0 does not meet any of the sets $\Sigma_{\rho_1}, \Sigma_{\rho_1'}, \Sigma_{\rho_1''}$. By Lemma 4.6 (iii) the closure of $\bigcup_{j=0}^{\infty} f^{-j}(W(p_1))$ contains T_1, and in particular ζ_1. Since U_0 is a one-sided

neighborhood of ζ_1, there exists $j_1 > 0$ such that $f^{j_1}(U_0) \cap W(p_1) \neq \emptyset$. In the covering space of the annulus, $f^{j_1}(U_0)$ intersects some copy $W^{h_1} := W(p_1) + (h_1, 0)$ of $W(p_1)$, where h_1 is some positive integer. Since U_0 does not intersect Σ_{ρ_1} and $\Sigma_{\rho'_1}$, it follows that $f^{j_1}(U_0)$ does not intersect Σ_{ρ_1} and $\Sigma_{\rho'_1}$. On the other hand, $f^{j_1}(U_0)$ intersects the essential circles $C_{\rho_1}, C_{\rho'_1}$, containing the Aubry-Mather sets $\Sigma_{\rho_1}, \Sigma_{\rho'_1}$, respectively.

Let $\gamma_1 : [0,1] \to U_0$ be a vertical curve, i.e., $\gamma_1(0) \in T_1$ and $\pi_x(\gamma_1(t)) = \pi_x(\gamma_1(0))$ for all t, such that $f^{j_1}(\gamma_1(1))$ is an intersection point of $f^{j_1}(U_0)$ with W^{h_1}. The curve $f^{j_1}(\gamma_1)$ is a positively tilted curve which crosses both essential circles C_{ρ_1} and $C_{\rho'_1}$. (See Remark 4.10.) Since $f^{j_1}(U_0)$ is disjoint from Σ_{ρ_1}, the intersections between $f^{j_1}(\gamma_1)$ and C_{ρ_1} occur within the 'gaps' of Σ_{ρ_1}, i.e., within the open interval components of $C_{\rho_1} \setminus \Sigma_{\rho_1}$.

We can assign an oriented intersection number between $f^{j_1}(\gamma_1)$ and each gap of Σ_{ρ_1}. Consider a homotopy $h_s : A \to A$, $s \in [0,1]$, such that $f^{j_1}(h_s(\gamma_1))$ keeps the endpoints of $f^{j_1}(\gamma_1)$ fixed for all s, $f^{j_1}(h_s(\gamma_1))$ does not intersect Σ_{ρ_1} for any $s \in [0,1]$, and $f^{j_1}(h_1(\gamma_1))$ is transverse to C_{ρ_1} (see [8]). We set the oriented intersection number of $f^{j_1}(h_1(\gamma_1))$ with C_{ρ_1} to be $+1$ at a point where the curve moves from below C_{ρ_1} to above C_{ρ_1} as t increases, and to be -1 at a point where the curve moves from above C_{ρ_1} to below C_{ρ_1} as t increases. Then we assign an oriented intersection number between $f^{j_1}(h_1(\gamma_1))$ and a gap of Σ_{ρ_1}, by adding the oriented intersection numbers for all of the intersection points within that gap. Since the oriented intersection number is preserved by homotopy, the oriented intersection number between $f^{j_1}(\gamma_1)$ and a gap is, by definition, the oriented intersection number between $f^{j_1}(h_1(\gamma_1))$ and that gap. Then the oriented intersection number between $f^{j_1}(\gamma_1)$ and C_{ρ_1} is the sum of the oriented intersection numbers over all gaps.

Since the curve $f^{j_1}(\gamma_1)$ starts from below C_{ρ_1} and ends above C_{ρ_1}, there exists a gap for which the oriented intersection number with $f^{j_1}(\gamma_1)$ is positive.

We follow the curve $t \mapsto f^{j_1}(\gamma_1(t))$ starting with $t = 0$ and we mark the first gap of Σ_{ρ_1} that is crossed by $f^{j_1}(\gamma_1)$ with a positive oriented intersection number; we denote by $a^1_{\rho_1}, b^1_{\rho_1}$ the endpoints of this gap. This means that if $f^{j_1}(\gamma_1)$ crosses other gaps of Σ_{ρ_1} that are to the left of this gap, it does so with 0 oriented intersection number. (If the first gap crossed would be crossed with negative oriented intersection number, it would violate the positive tilt condition of $f^{j_1}(\gamma_1)$.) Thus, when the curve $f^{j_1}(\gamma_1)$ crosses the gap between $a^1_{\rho_1}, b^1_{\rho_1}$, it comes from below the circle C_{ρ_1}. Following the curve segment of $f^{j_1}(\gamma_1(t))$ after crossing the gap of endpoints $a^1_{\rho_1}, b^1_{\rho_1}$, we mark the first gap of $\Sigma_{\rho'_1}$ that is crossed by $f^{j_1}(\gamma_1)$ with positive oriented intersection number, and we denote by $a^1_{\rho'_1}, b^1_{\rho'_1}$ its endpoints. Similarly, when the curve $f^{j_1}(\gamma_1)$ crosses the gap between $a^1_{\rho'_1}, b^1_{\rho'_1}$, it comes from below the circle $C_{\rho'_1}$.

We claim that the left endpoint of the lower gap is to the right of the right endpoint of the upper gap, i.e., $\pi_x(a^1_{\rho_1}) < \pi_x(b^1_{\rho'_1})$. Otherwise, if $\pi_x(a^1_{\rho_1}) \geq \pi_x(b^1_{\rho'_1})$, then there exists an arc $f^{j_1}(\gamma_1(s))$, $s \in [s_1, s_2]$, such that $\pi_x(f^{j_1}(\gamma_1(s_1))) = \pi_x(f^{j_1}(\gamma_1(s_2))) = \pi_x(a^1_{\rho_1})$, $\pi_y(f^{j_1}(\gamma_1(s_1))) < \pi_y(f^{j_1}(\gamma_1(s_2)))$, and $\pi_x(f^{j_1}(\gamma_1(s))) \geq \pi_x(a^1_{\rho_1})$ for all $s \in (s_1, s_2)$. This implies that either the angle deviation from the vertical $\theta(s)$ along the curve $f^{j_1}(\gamma_1)$ becomes non-positive for some $s \in (s_1, s_2)$, or that there exists another arc $f^{j_1}(\gamma_1(\tau))$, $\tau \in [\tau_1, \tau_2]$, with $\tau_1 < \tau_2 < s_1 < s_2$, such that $\pi_x(f^{j_1}(\gamma_1(\tau_1))) = \pi_x(f^{j_1}(\gamma_1(\tau_2))) = \pi_x(a^1_{\rho_1})$, $\pi_y(f^{j_1}(\gamma_1(\tau_1))) > \pi_y(f^{j_1}(\gamma_1(\tau_2)))$,
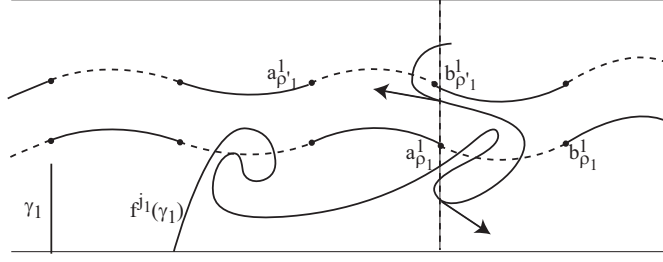
FIGURE 4. Violation of the positive tilt condition.

and $\pi_x(f^{j_1}(\gamma_1(s))) \leq \pi_x(f^{j_1}(\gamma_1(\tau)))$ for all $s \in (s_1, s_2)$ and $\tau \in (\tau_1, \tau_2)$. In the first case we obtain a contradiction with the positive tilt condition on the curve $f^{j_1}(\gamma_1)$. See Fig. 4. In the second case we obtain a contradiction with the fact that $f^{j_1}(\gamma_1)$ comes from below C_{ρ_1} before crossing the gap of endpoints $a^1_{\rho_1}, b^1_{\rho_1}$. The conclusion of this step is that the curve $f^{j_1}(\gamma_1(t))$ passes through the gap between $a^1_{\rho_1}$ and $b^1_{\rho_1}$ of Σ_{ρ_1}, from below C_{ρ_1} to above C_{ρ_1}, then it passes through the gap between $a^1_{\rho'_1}$ and $b^1_{\rho'_1}$ of $\Sigma_{\rho'_1}$, from below $C_{\rho'_1}$ to above $C_{\rho'_1}$, and $\pi_x(a^1_{\rho_1}) < \pi_x(b^1_{\rho'_1})$.

Now we consider a one-sided rectangular neighborhood $U_1 \subseteq U_0$ of some point in T_1, bounded below by T_1, to the left by γ_1, and to the right by some other vertical curve segment γ'_1. If γ'_1 is sufficiently close to γ_1, then, by continuity, the image of each vertical curve in U_1 under f^{j_1} crosses the gap between $a^1_{\rho_1}$ and $b^1_{\rho_1}$ of Σ_{ρ_1} with positive oriented intersection number, and crosses the gap between $a^1_{\rho'_1}$ and $b^1_{\rho'_1}$ of $\Sigma_{\rho'_1}$ with positive oriented intersection number. We choose and fix a set $U_1 \subseteq U_0$ with these properties. By Lemma 4.6 (iii) (see also Remark 4.9) the closure of $\bigcup_{j=0}^{\infty} f^{-j}(W(p_1))$ contains T_1, so there exists $j_2 > j_1$ such that, in the annulus, $f^{j_2}(U_1) \cap W(p_1) \neq \emptyset$, and, in the covering space of the annulus, $f^{j_2}(U_1)$ intersects some copy $W^{h_2} := W(p_1) + (h_2, 0)$ of $W(p_1)$ for some positive integer $h_2 > h_1$. (Due to the positive twist condition on f and the assumption that the rotation numbers on the boundary components of the annulus are positive, the vertical line $\{x = \pi_x(p_1) + h_1\}$ is mapped by $f^{j_2-j_1}$ to a positive tilted map to the right of $\{x = \pi_x(p_1) + h_1\}$, hence, if j_2 is large enough, we can choose $h_2 > h_1$.)

Then there exists a vertical curve $\gamma_2 : [0, 1] \to U_1$, such that $f^{j_2}(\gamma_2(1))$ is an intersection point of $f^{j_2}(U_1)$ with W^{h_2}. The curve $f^{j_2}(\gamma_2(t))$ crosses C_{ρ_1} and $C_{\rho'_1}$. Let $a^2_{\rho_1}, b^2_{\rho_1}$ be the endpoints of the leftmost gap of Σ_{ρ_1} that is crossed by $f^{j_2}(\gamma_2(t))$ with positive oriented intersection number equal, and let $a^2_{\rho'_1}, b^2_{\rho'_1}$ be the endpoints of the leftmost gap of $C_{\rho'_1}$ that is crossed by $f^{j_2}(\gamma_2(t))$ with positive oriented intersection number. The image curve $f^{j_2}(\gamma_2)$ is a positively tilted curve located on the 'right side' of the positively tilted curve $f^{j_2}(\gamma_1)$, in the sense that any graph over x that intersects both $f^{j_2}(\gamma_1)$ and $f^{j_2}(\gamma_2)$ has the leftmost intersection point with the $f^{j_1}(\gamma_1)$. Therefore, the gap endpoints $a^2_{\rho_1}, b^2_{\rho_1}$ are either the image under $f^{j_2-j_1}$ of the gap endpoints $a^1_{\rho_1}, b^1_{\rho_1}$ found at the previous step, or are the image under $f^{j_2-j_1}$ of some other gap endpoints of Σ_{ρ_1} located to the right of the gap between $a^1_{\rho_1}$ and $b^1_{\rho_1}$. Similarly, the gap endpoints $a^2_{\rho'_1}, b^2_{\rho'_1}$ are either the image under $f^{j_2-j_1}$ of the gap endpoints $a^1_{\rho'_1}, b^1_{\rho'_1}$ from the previous step, or are the image

under $f^{j_2-j_1}$ of some other gap of $\Sigma_{\rho_1'}$ located to the right of the gap between $a_{\rho_1'}^1$ and $b_{\rho_1'}^1$.

Then, there exists a one-sided rectangular neighborhood $U_2 \subseteq U_1$ of some point in T_1, bounded below by T_1, to the left by γ_2, and to the right by some other vertical curve segment γ_2', such that the image of each vertical curve in U_2 under f^{j_2} crosses the gap between $a_{\rho_1}^2$ and $b_{\rho_1}^2$ of Σ_{ρ_2} with positive oriented intersection number, and crosses the gap between $a_{\rho_1'}^2$ and $b_{\rho_1'}^2$ of $\Sigma_{\rho_1'}$ with positive oriented intersection number.

Recursively, we obtain a nested sequence of one-sided neighborhoods of points in T_1, denoted $U_1 \supseteq U_2 \supseteq \ldots U_m \supseteq \ldots$, all contained in U_0, and two sequences of positive integers $j_1 < j_2 < \ldots < j_m < \ldots$ and $h_1 < h_2 < \ldots < h_m < \ldots$ with the following properties:

(i) each set U_m is a topological rectangle consisting of vertical curves starting from T_1, bounded on the left-side by a vertical curve γ_m and on the right by a vertical curve γ_m';

(ii) $f^{j_m}(U_m) \cap W^{h_m} \neq \emptyset$, where $W^{h_m} := W(p_1) + (h_m, 0)$;

(iii) the image of each vertical curve in U_m under f^{j_m} crosses C_{ρ_1} with positive oriented intersection number through a gap between $a_{\rho_1}^m$ and $b_{\rho_1}^m$ of Σ_{ρ_1}, and it crosses $C_{\rho_1'}$ with positive oriented intersection number through a gap between $a_{\rho_1'}^m$ and $b_{\rho_1'}^m$ of $\Sigma_{\rho_1'}$;

(iv) $\pi_x(a_{\rho_1}^m) < \pi_x(b_{\rho_1'}^m)$;

(v) the gap endpoints $a_{\rho_1}^m, b_{\rho_1}^m$ are the images under $f^{j_m-j_{m-1}}$ of the gap endpoints $a_{\rho_1}^{m-1}, b_{\rho_1}^{m-1}$, or of the endpoints of some other gap in Σ_{ρ_1} located to the right side of this gap; the gap endpoints $a_{\rho_1'}^m, b_{\rho_1'}^m$ are the images under $f^{j_m-j_{m-1}}$ of the gap endpoints $a_{\rho_1'}^{m-1}, b_{\rho_1'}^{m-1}$, or of the endpoints of some other gap in $\Sigma_{\rho_1'}$ located to the right side of this gap.

The endpoints of a gap of Σ_{ρ_1} or $\Sigma_{\rho_1'}$ are mapped by f into the endpoints of some other gap of Σ_{ρ_1} or $\Sigma_{\rho_1'}$, respectively. Also, the order of the gaps is preserved under iteration. The endpoints of the gap in Σ_{ρ_1} are iterated with rotation number ρ_1, and the endpoints of the gap in $\Sigma_{\rho_1'}$ are iterated with rotation number $\rho_1' > \rho_1$. Then, for some sufficiently large iterate j_m the order of the gaps gets reversed in the annulus. That is, we get the following ordering in terms of the angle coordinate in the covering space of the annulus:

(i) $\pi_x(a_{\rho_1}^m) < \pi_x(b_{\rho_1}^m) < \pi_x(a_{\rho_1}^1) + h_m < \pi_x(b_{\rho_1}^1) + h_m$,

(ii) $\pi_x(a_{\rho_1'}^1) + h_m < \pi_x(b_{\rho_1'}^1) + h_m < \pi_x(a_{\rho_1'}^m) < \pi_x(b_{\rho_1'}^m)$.

Since $f^{j_1}(U_1)$ and $f^{j_m}(U_m)$ are connected, the above ordering of the crossings with C_{ρ_1} and $C_{\rho_1'}$ implies that $f^{j_1}(U_1)$ and $f^{j_m}(U_m)$ have an intersection point above C_{ρ_1}. As $U_1, U_m \subseteq U_0$, it follows that $f^{j_1}(U_0)$ and $f^{j_m}(U_0)$ have an intersection point above C_{ρ_1}. Since $f^{j_1}(U_0)$ and $f^{j_m}(U_0)$ are connected and disjoint from the Aubry-Mather set $\Sigma_{\rho_1} \subseteq C_{\rho_1}$, there exists a component $\mathcal{U}$ of the complement $\mathcal{Z} \setminus [f^{j_1}(U_0) \cup f^{j_m}(U_0)]$ which is an open topological disk containing some point $\xi_1 \in \Sigma_{\rho_1}$. The boundary of $\mathcal{U}$ consists of a finite union of sub-arcs of the boundaries of $f^{j_1}(U_0)$ and $f^{j_m}(U_0)$, and possibly of curve segments of T_1. See Figure 5. Letting $j_* = j_1$ and $j_*' = j_m$ ends the proof of the claim.

We now apply the statement of the claim to T_2, starting with the one-sided neighborhood V_0 of $\zeta_2 \in T_2$. We choose three Aubry-Mather sets $\Sigma_{\rho_2}, \Sigma_{\rho_2'}, \Sigma_{\rho_2''}$
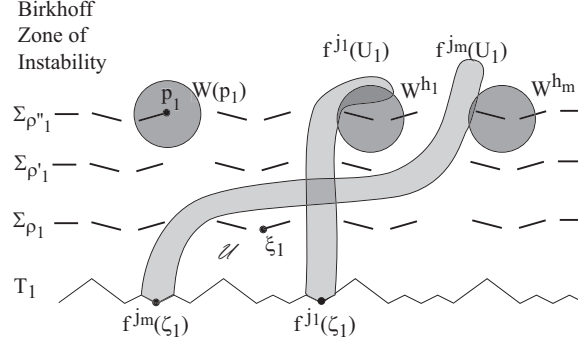
FIGURE 5. An open topological disk forming a neighborhood of a point in the Aubry-Mather set Σ_{ρ_1}.

in $\mathcal{Z}$, lying on three essential circles $C_{\rho_2}, C_{\rho'_2}, C_{\rho''_2}$, respectively, with $\rho_2 > \rho'_2 > \rho''_2$ irrational rotation numbers, and $C_{\rho_2} \succ C_{\rho'_2} \succ C_{\rho''_2}$.

The statement in the claim implies that there exists a neighborhood $\mathcal{V}$ of some point $\xi_2 \in \Sigma_{\rho_2}$, homoeomorphic to an open disk, whose boundary consists of a finite union of sub-arcs in the boundaries of $f^{-l_*}(V_0)$ and $f^{-l'_*}(V_0)$, for some $l_* < l'_*$, and possibly a finite union of curve segments of T_2.

By Theorem 4.7 there is an orbit that goes from $\mathcal{U}$ to $\mathcal{V}$, i.e., there exists k_* such that $f^{k_*}(\mathcal{U}) \cap \mathcal{V} \neq \emptyset$. The boundary of $\mathcal{U}$ (resp. $\mathcal{V}$) is a simple closed curved separating the annulus into two connected components. Also, the boundary of each of $f^{j_*}(U_0), f^{j'_*}(U_0), f^{-l_*}(V_0), f^{-l'_*}(V_0)$ is a simple closed curve. Since $f^{k_*}(\mathcal{U}) \cap \mathcal{V} \neq \emptyset$, the Jordan Curve Theorem implies that either $\partial f^{k_*}(\mathcal{U}) \cap \mathrm{int}[f^{-l_*}(V_0) \cup f^{-l'_*}(V_0)] \neq \emptyset$ or $\partial \mathcal{V} \cap \mathrm{int}[f^{j_*}(U_0) \cup f^{j'_*}(U_0)] \neq \emptyset$.

It follows that $\mathrm{int}(f^{k_*+j_*}(U_0))$ or $\mathrm{int}(f^{k_*+j'_*}(U_0))$ has a non-empty intersection with $\mathrm{int}(f^{-l_*}(V_0))$ or $\mathrm{int}(f^{-l'_*}(V_0))$. Thus, some forward iterate of $\mathrm{int}(U_0)$ intersects $\mathrm{int}(V_0)$. Hence there exists $N > 0$ such that $f^N(\mathrm{int}(U_0)) \cap \mathrm{int}(V_0) \neq \emptyset$. Since the sets $f^N(U_0)$ and V_0 are topological disks have interior points in common, but also points that are not in common (namely the points lying on T_1 and T_2, respectively), the Jordan Curve Theorem implies that $f^N(\partial U_0) \cap \partial V_0 \neq \emptyset$.

The remaining case of the proof, when the interior of U_0 does intersect some Aubry-Mather set and the interior of V_0 does not, or when the interior of U_0 does intersect some Aubry-Mather set and the interior of V_0 does not, follows easily from the above arguments. $\square$

The next statement says that given two points on the boundary tori of a BZI, and a finite sequence of Aubry-Mather sets inside the zone, there exists an orbit that starts in a prescribed neighborhood of the point on the lower boundary torus, then moves on and shadows, in the sense of the ordering of the orbit, each Aubry-Mather set in the sequence, and ends in a prescribed neighborhood of the point on the upper boundary torus. This result extends Theorem 4.8, and relies on the topological argument of Hall. As in the previous theorem, we do not need any extra conditions on the dynamics on the boundary tori. The resulting shadowing orbits are not necessarily minimal.

Theorem 6.2. *Suppose that T_1 and T_2 bound a BZI $\mathcal{Z}$. Let $\zeta_1 \in T_1, \zeta_2 \in T_2$, U be a neighborhood of ζ_1, and V a neighborhood of ζ_2. Let $\{\Sigma_{\omega_s}\}_{s \in \{1,\dots,\sigma\}}$ be a finite collection of Aubry-Mather sets inside $\mathcal{Z}$ such that each Σ_{ω_s} lies on some essential circle C_{ω_s} that is a graph over the x-coordinate, with $C_{\omega_s} \prec C_{\omega_{s'}}$, provided $\omega_s < \omega_{s'}$. Let $\{n_s\}_{s=1,\dots,\sigma}$ be sequence of positive integers. Then there exist a point $z \in U$, and a sequence of positive integers $\{m_s\}_{s=0,\dots,\sigma}$, such that, for each $s \in \{1,\dots,\sigma\}$,*

$$\pi_x(f^j(w_s)) < \pi_x(f^j(z)) < \pi_x(f^j(\bar{w}_s)) \text{ for}$$

(6.1)
$$\sum_{t=1}^{s-1} n_t + \sum_{t=0}^{s-1} m_t \leq j \leq \sum_{t=1}^{s} n_t + \sum_{t=0}^{s-1} m_t,$$

where w_s and $\bar{w}_s$ are some points in the Aubry-Mather set Σ_{ω_s}, and $f^N(z) \in V$ for $N = \sum_{t=1}^{\sigma} n_t + \sum_{t=0}^{\sigma} m_t$. The number N can be chosen arbitrarily large.

Moreover, if U, V are chosen so that $U \cap \mathcal{Z}, V \cap \mathcal{Z}$ are topological disks, then there exists a point $z' \in \partial U$ satisfying the ordering condition (6.1) such that $f^N(z') \in \partial V$.

Proof. We use the construction of diagonal sets described in the sketch of the proof of Theorem 4.8; for details see [29].

Part 1. As in the proof of Theorem 6.1 we can choose one sided compact neighborhoods U_0 of $\zeta_1 \in T_1$ and V_0 of $\zeta_2 \in T_2$, such that $U_0 \cap \mathcal{Z}$ and $V_0 \cap \mathcal{Z}$ are topological disks.

We first prove the existence of a point $z \in U_0$ satisfying (6.1) and such that $f^N(z) \in V_0$. Let C_{ω_1} be the essential circle containing Σ_{ω_1}. We choose an Aubry-Mather set Σ_{ρ_1} lying on some essential circle C_{ρ_1}, such that $C_{\omega_1} \prec C_{\rho_1}$. We choose a point $p_1 \in \Sigma_{\rho_1}$ and a small neighborhood $W(p_1)$ of p_1 which does not intersect Σ_{ω_1}. We assume that the interior of U_0 does not meet Σ_{ω_1} and Σ_{ρ_1}, otherwise the proof follows as in [29]. By Lemma 4.6 (iii) the closure of $\bigcup_{j=0}^{\infty} f^{-j}(W(p_1))$ contains T_1, and in particular ζ_1. Proceeding as in the proof of Theorem 6.1, we obtain a nested sequence $U_1 \supseteq U_2 \supseteq \dots \supseteq U_i$ of one-sided neighborhoods of points in T_1, all contained in U_0, and two sequences of positive integers $j_1 < j_2 < \dots < j_i < \dots$ and $h_1 < h_2 < \dots < h_i < \dots$ with the following properties:

(i) each set U_i is a topological rectangle consisting of vertical curves starting from T_1, bounded on the left-side by a vertical curve γ_i and on the right by a vertical curve γ_i';

(ii) $f^{j_i}(U_i) \cap W^{h_i} \neq \emptyset$, where $W^{h_i} := W(p_1) + (h_i, 0)$;

(iii) the image of each vertical curve in U_i under f^{j_i} crosses C_{ω_1} with positive oriented intersection number through a gap in Σ_{ω_1} of endpoints $a_{\omega_1}^i$ and $b_{\omega_1}^i$; the gap is chosen to be the first gap that is crossed over with positive oriented intersection number;

(iv) the endpoints of the gap between $a_{\omega_1}^i$ and $b_{\omega_1}^i$ are the images under $f^{j_i-j_{i-1}}$ of either the endpoints of the gap between $a_{\omega_1}^{i-1}$ and $b_{\omega_1}^{i-1}$, or of a gap in Σ_{ω_1} located to the right side of that gap.

Since the rotation number of Σ_{ω_1} is smaller than the rotation number of Σ_{ρ_1}, any pair of points chosen on these two sets shift apart from one another under positive iterations. Therefore there exists some i large enough so that the gap of endpoints $a_{\omega_1}^i$ and $b_{\omega_1}^i$ is on the left side of W^{h_i}, in the sense that $\pi_x(b_{\omega_1}^i) < \pi_x(z)$ for all $z \in W^{h_i}$.

We claim that, by choosing i large enough and γ_i' sufficiently close to γ_i, we can ensure that the set $f^{j_i}(U_i)$ has a part which is a positive diagonal set in $B_{a_{\omega_1}^i, b_{\omega_1}^i}$.
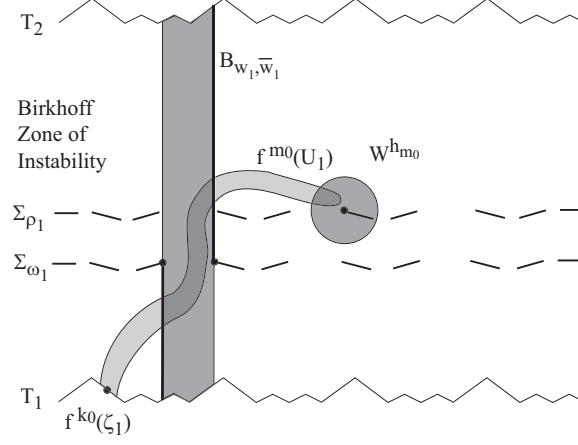
FIGURE 6. A positive diagonal set.

Now we justify the claim. The image of the left-side γ_i of U_i is mapped by f^{j_i} onto a positively tilted curve that crosses the gap between $a^i_{\omega_1}$ and $b^i_{\omega_1}$ with positive oriented intersection number. Cutting the curve $f^{j_1}(\gamma_1)$ with the vertical strip $B_{a^i_{\omega_1}, b^i_{\omega_1}}$ yields at least one component that connects $I_{a^i_{\omega_1}} = \{x = a^i_{\omega_1}\}$ to $I_{b^i_{\omega_1}} = \{x = b^i_{\omega_1}\}$ with positive oriented intersection number. Take the first such a component and follow it in the direction of the increase of the parameter t. The intersection of this component with $I_{a^i_{\omega_1}}$ needs to occur at a point r_1 below $a^i_{\omega_1}$, i.e. $r_1 \in I^-_{a^i_{\omega_1}}$, otherwise this component does not come from below C_{ω_1}. Following this component forward starting from r_1, the curve cannot intersect $I_{a^i_{\omega_1}}$ above $a^i_{\omega_1}$ as this would violate the positive tilt condition, or the choice of the gap between $a^i_{\omega_1}$ and $b^i_{\omega_1}$ being the first gap that is crossed with positive oriented intersection number. Following the component starting from r_1, it must first meet $I_{b^i_{\omega_1}}$ at a point r'_1 above $b^i_{\omega_1}$, i.e. $r'_1 \in I^+_{b^i_{\omega_1}}$; otherwise this component does not have positive oriented intersection number with the gap between $a^i_{\omega_1}$ and $b^i_{\omega_1}$. Therefore, the component of $f^{j_i}(\gamma_i)$ between r_1 and r'_1 goes from $I^-_{a^i_{\omega_1}}$ to $I^+_{b^i_{\omega_1}}$ without intersecting again $I^+_{a^i_{\omega_1}}$ or $I^-_{b^i_{\omega_1}}$. Now taking a curve γ'_i sufficiently close to γ_i results in a set U_i with the property that $f^{j_i}(U_i)$ has a part which is a positive diagonal set in $B_{a^i_{\omega_1}, b^i_{\omega_1}}$.

We change notation at this point: we denote $m_0 := j_i$, $w_1 := a^i_{\omega_1}$, and $\bar{w}_1 := b^i_{\omega_1}$, $U' = U_i$ for i fixed as above. Thus, the points w_1 and $\bar{w}_1$ are the endpoints of a gap in Σ_{ω_1}, and $f^{m_0}(U')$ has a part which is a positive diagonal in $B_{w_1, \bar{w}_1}$. The positive integer m_0 is the first term of the sequence $\{m_s\}_{s=0,\ldots,\sigma}$ in the statement of the theorem. Note that U' consists of a union of vertical segments emerging from T_1. See Figure 6.

Using the construction described in the sketch of the proof of Theorem 4.8, we obtain a nested sequence $D_0 \supseteq D_1 \supseteq \ldots \supseteq D_\sigma$ of negative diagonals of $B_{w_1, \bar{w}_1}$ and a sequence of positive integers $\{m_s\}_{s=0,\ldots,\sigma-1}$ such that for each $s \in \{1, \ldots, \sigma\}$ and

each $z \in D_s$ we have

(6.2) $\pi_x(f^j(w_s)) < \pi_x(f^j(z)) < \pi_x(f^j(\bar{w}_s))$ for $j_s \leq j \leq j_s + n_s$,

where $j_s := \sum_{t=1}^{s} n_t + \sum_{t=0}^{s-1} m_t$, w_s and $\bar{w}_s$ are the endpoints of some gap in the Aubry-Mather set Σ_{ω_s}, and $f^{(j_s+n_s)}(D_s)$ is a positive diagonal in $B_{f^{n_s}(w_s), f^{n_s}(\bar{w}_s)}$. In particular, $f^{(j_\sigma+n_\sigma)}(D_\sigma)$ is a positive diagonal in $B_{f^{n_\sigma}(w_\sigma), f^{n_\sigma}(\bar{w}_\sigma)}$, where w_σ and $\bar{w}_\sigma$ are the endpoints of some gap in the Aubry-Mather set Σ_{ω_σ}.

Since $f^{m_0}(U')$ has a part which is a positive diagonal in $B_{w_1, \bar{w}_1}$ and D_σ is a negative diagonal of $B_{w_1, \bar{w}_1}$, then $f^{m_0}(U')$ and D_σ have a non-empty intersection. Also, $f^{(j_\sigma+n_\sigma)}(U')$ has a part which is a positive diagonal in $B_{f^{n_\sigma}(w_\sigma), f^{n_\sigma}(\bar{w}_\sigma)}$ that stretches across $f^{(j_\sigma+n_\sigma)}(D_\sigma)$.

Now we start with the one-sided neighborhood V_0 of $\zeta_2 \in T_2$. Let C_{ω_σ} be an essential circle containing Σ_{ω_σ}. We choose an Aubry-Mather set Σ_{ρ_2} lying on an essential circle C_{ρ_2}, such that C_{ρ_2} is below C_{ω_σ}. We assume that the interior of V_0 does not meet Σ_{ω_σ} and Σ_{ρ_2}, otherwise the proof follows as in [29]. Using Lemma 4.6 (iii) and following the procedure described above for negative iterations, we produce a one-sided neighborhood V' of a point in T_2, with $V' \subseteq V_0$, and a positive integer m'_σ such that $f^{-m'_\sigma}(V')$ contains a part which is a negative diagonal in $B_{w'_\sigma, \bar{w}'_\sigma}$, where w'_σ and $\bar{w}'_\sigma$ are the endpoints of a gap in Σ_{ω_σ}.

By using the existence of orbits passing from near T_2 to near T_1, and of orbits passing from near T_1 to near T_2, as in the sketch of the proof of Theorem 4.8, we can further iterate the positive diagonal $f^{(j_\sigma+n_\sigma)}(U')$ from above, so that we obtain an iterate $f^{(j_\sigma+n_\sigma+m''_\sigma)}(U')$ which contains a component that stretches all the way across a fundamental interval of the annulus. In particular, $f^{(j_\sigma+n_\sigma+m''_\sigma)}(U')$ contains a component that is a positive diagonal of $B_{w'_\sigma, \bar{w}'_\sigma}$. Since $f^{-m'_\sigma}(V')$ is a negative diagonal in $B_{w'_\sigma, \bar{w}'_\sigma}$, then $f^{(j_\sigma+n_\sigma+m''_\sigma)}(U')$ has a nonempty intersection with $f^{-m'_\sigma}(V')$. Equivalently, $f^{(j_\sigma+n_\sigma+m_\sigma)}(U')$ has a non-empty intersection with V', where $m_\sigma := m'_\sigma + m''_\sigma$.

Thus, each point $z \in U' \cap f^{-(j_\sigma+n_\sigma+m_\sigma)}(V')$ goes from the neighborhood U_0 of ζ_1 to the neighborhood V_0 of ζ_2 and it shadows, in the sense of the ordering, each of the Aubry-Mather set Σ_{ω_s}, $s = 1, \ldots, \sigma$, along the way.

Part 2. Now we explain how to modify the above proof to show that there exists a point $z' \in \partial U_0$ that satisfies (6.1) and $f^N(z') \in \partial V_0$. This does not follow immediately from the above argument since the image of ∂U_0 under iteration may fail being a positively tilted curve; hence we cannot infer that $f^{m_0}(\partial U_0)$ intersects the negative diagonal set D_σ from above.

By Theorem 6.1, there exist $l > 0$ and a point $q \in U_0$ depending on l such that $f^l(q)$ is in some prescribed neighborhood of a point $r \in T_2$, where l can be chosen arbitrarily large. Since the points of T_1 and T_2 have different rotation numbers hence move apart under iteration, there exists l_0 sufficiently large such that $f^{l_0}(T_1 \cap U_0)$ and $f^{l_0}(q)$ are separated by a fundamental interval of the annulus, i.e., $\pi_x(f^{l_0}(r)) - \pi_x(f^{l_0}(q)) > 1$ for all $r \in T_1 \cap U_0$. Let $x_0, \bar{x}_0 \in \mathbb{R}$ be such that $\pi_x(f^{l_0}(r)) < x_0 < \bar{x}_0 < \pi_x(f^{l_0}(q))$ and $1 < \bar{x}_0 - x_0$. Let w_0 be a point on the vertical line $\{x = x_0\}$ whose y-coordinate is larger than that of any point in $f^{l_0}(U_0) \cap \{x = x_0\}$. Similarly, let $\bar{w}_0$ be a point on the vertical line $\{x = \bar{x}_0\}$ whose y-coordinate is smaller than that of any point in $f^{l_0}(U_0) \cap \{x = \bar{x}_0\}$. Then $f^{l_0}(U_0) \cap \text{cl}(B_{w_0, \bar{w}_0})$ has a component that is a positive diagonal in $B_{w_0, \bar{w}_0}$.
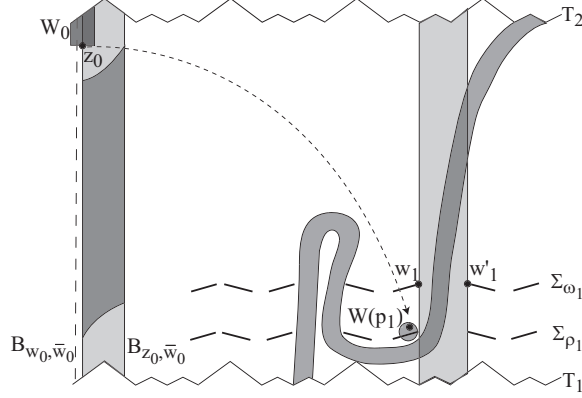
FIGURE 7. Construction of a positive diagonal set.

Let Σ_{ω_1} be the first set in the prescribed sequence of Aubry-Mather sets. Now we want to show that, by slightly adjusting the vertical strip $B_{w_0,\bar{w}_0}$ to a new vertical strip $B_{z_0,\bar{z}_0}$, there exists a diagonal set in $f^{l_0}(U_0) \cap B_{z_0,\bar{z}_0}$ such that a sufficiently large iterate of this diagonal set has a component that is a positive diagonal in $B_{w_1,\bar{w}_1}$, for some points $w_1, \bar{w}_1 \in \Sigma_{\omega_1}$.

There exists x_0' sufficiently close to x_0 such that for all x_0'' between x_0 and x_0', the point $(x_0'', \pi_y(w_0))$ has the y-coordinate larger than that of any point in $f^{l_0}(\mathrm{cl}(U_0)) \cap \{x = x_0''\}$. Then the set $W_0 = \{(x_0'', y_0'') \,|\, x_0 < x_0'' < x_0', \pi_y(w_0) < y_0''\}$ is a neighborhood of an arc in T_2, with the property that each point $(x_0'', y_0'') \in W_0$ has the y-coordinate larger than that of any point in $f^{l_0}(\mathrm{cl}(U_0)) \cap \{x = x_0''\}$. Similarly, there is $\bar{x}_0'$ sufficiently close to $\bar{x}_0$ such that for all $\bar{x}_0''$ between $\bar{x}_0$ and $\bar{x}_0'$, the point $(\bar{x}_0'', \pi_y(\bar{w}_0))$ has the y-coordinate smaller than that of any point in $f^{l_0}(\mathrm{cl}(U_0)) \cap \{x = \bar{x}_0''\}$. Then the set $\bar{W}_0 = \{(\bar{x}_0'', \bar{y}_0'') \,|\, \bar{x}_0 < \bar{x}_0'' < \bar{x}_0', \pi_y(\bar{w}_0) > \bar{y}_0''\}$ is a neighborhood of an arc in T_1, with the property that each point $(\bar{x}_0'', \bar{y}_0'') \in \bar{W}_0$ has the y-coordinate smaller than that of any point in $f^{l_0}(\mathrm{cl}(U_0)) \cap \{x = \bar{x}_0''\}$.

Let Σ_{ρ_1} be an Aubry-Mather set lying on an essential circle C_{ρ_1} that is below the essential circle C_{ω_1} containing Σ_{ω_1}, let $p_1 \in C_{\rho_1}$, and let $W(p_1)$ be a small neighborhood of p_1 that does not intersect Σ_{ω_1}. Then there exists $z_0 \in W_0$ and j_0 sufficiently large such that $f^{j_0}(z_0) \in W(p_1)$. Since the curve $f^{j_0}(I_{z_0}^+)$ is a positively tilted curve emerging from T_2, the arguments used in Part 1 show that there is a gap of the Aubry-Mather set Σ_{ω_1}, between a pair of points $w_1, w_1' \in \Sigma_{\omega_1}$, such that $f^{j_0}(I_{z_0}^+)$ crosses this gap with negative intersection number (where the parametrization of $f^{j_0}(I_{z_0}^+)$ is chosen so that to $t = 0$ it corresponds a point on T_2) and has its first intersection with I_{w_1} below the point w_1, provided j_0 is chosen large enough. This implies that the image of the diagonal component of $f^{l_0}(\mathrm{cl}(U_0)) \cap B_{z_0,\bar{w}_0}$ under f^{j_0} has a component Δ_0 in $B_{w_1,w_1'}$ that satisfies the positive diagonal set conditions relative to its left side. See Fig. 7.

Now, for all $j' > 0$, the image of Δ_0 under $f^{j'}$ also has a component that satisfies the positive diagonal set conditions relative to the left side of $B_{f^{j'}(w_1), f^{j'}(w_1')}$. (This follows from the hereditary property of diagonal sets, see Section 4.) In a similar fashion, there exist a point $\bar{z}_0 \in \bar{W}_0$ and j_0' sufficiently large such that the positively tilted curve $f^{j_0'+j_0}(I_{\bar{z}_0}^-)$, emerging from T_1, crosses a gap of the Aubry-Mather

set Σ_{ω_1} between a pair of points $\bar{w}_1, \bar{w}_1' \in \Sigma_{\omega_1}$, the oriented intersection number between $f^{j_0'+j_0}(I_{\bar{z}_0}^-)$ and this gap is positive, and the first intersection with $I_{\bar{w}_1'}$ occurs above the point $\bar{w}_1'$. Thus, the image of Δ_0 under $f^{j_0'}$ has a component that satisfies the positive diagonal set conditions relative to its right side.

In summary, the image of the diagonal set component of $f^{l_0}(\mathrm{cl}(U_0)) \cap B_{z_0,\bar{z}_0}$ under $f^{j_0+j_0'}$ contains a component Δ_0' that is a positive diagonal in $B_{w_1,\bar{w}_1'}$, where $w_1, \bar{w}_1'$ are two points in the Aubry-Mather set Σ_{ω_1}. Moreover, the upper edge and the lower edge of Δ_0' are contained in $f^{j_0'+j_0+l_0}(\partial U_0)$.

We apply an analogous argument at the other boundary torus T_2. Given V_0 a neighborhood of a point $\zeta_2 \in T_2$, there exist $L > 0$ and a pair of points $w_\sigma, \bar{w}_\sigma \in \Sigma_{\omega_\sigma}$ such that $f^{-L}(\mathrm{cl}(V_0)) \cap B_{w_\sigma,\bar{w}_\sigma}$ has a component D_σ'' that is a negative diagonal in $B_{w_\sigma,\bar{w}_\sigma}$. The upper edge and the lower edge of this positive diagonal set are contained in $f^{-L}(\partial V_0)$.

Now we apply the argument from Part 1. There exists a negative diagonal set D_σ in $B_{w_1,\bar{w}_1}$ such that all points $z \in D_\sigma$ satisfy (6.1). By the argument for Theorem 4.8, the upper and lower edge of D_σ lie on $f^{-j_\sigma}(I_{w_\sigma}^-), f^{-j_\sigma}(I_{\bar{w}_\sigma}^+)$ where $j_\sigma = \sum_{s=0}^{\sigma} n_s + \sum_{s=0}^{\sigma-1} m_s$. Since negative and positive diagonal sets in the same vertical strip always intersect, the negative diagonal set D_σ intersects Δ_0', and in particular it intersects its upper and lower edges that are contained in $f^{j_0'+j_0+l_0}(\partial U_0)$. Iterating Δ_0' forward for j_σ times yields a positive diagonal set D_σ' in $B_{w_\sigma,\bar{w}_\sigma}$. The upper and lower edges of D_σ' are contained in $f^{j_\sigma+j_0'+j_0+l_0}(\partial U_0)$. The positive diagonal set D_σ' intersects the negative diagonal component D_σ'' of $f^{-L}(\mathrm{cl}(V_0)) \cap B_{w_\sigma,\bar{w}_\sigma}$. In particular the upper and lower edges of D_σ', that are contained in $f^{j_\sigma+j_0'+j_0+l_0}(\partial U_0)$, intersect the upper and lower edges of D_σ'' that are contained in $f^{-L}(\partial V_0)$.

Thus, there exists a point $z' \in \partial U_0$ that is taken by $f^{j_\sigma+j_0'+j_0+l_0}$ to ∂V_0 and satisfies the ordering relations (6.1). $\square$

7. A SHADOWING LEMMA IN NORMALLY HYPERBOLIC INVARIANT MANIFOLDS

In this section we present a shadowing lemma-type of result saying that, given a sequence of windows within a normally hyperbolic invariant manifold, consisting of pairs of windows correctly aligned under the scattering map, alternating with pairs of windows correctly aligned under some iterate of the inner map, then there exists a true orbit in the full space dynamics that follows these windows. This result reduces the construction of windows within the full dimensional phase space to the construction of lower dimensional windows within the normally hyperbolic invariant manifold.

For this section, we assume a diffeomorphism $f : M \to M$ on a manifold M, and an l-dimensional normally hyperbolic invariant manifold $\Lambda \subseteq M$ as in Subsection 5.1.

In the subsequent sections, we will apply Lemma 7.1 only in the case when Λ is a 2-dimensional normally hyperbolic invariant manifold in M, i.e., $l = 2$. A more general version of this lemma which does not involve the scattering map, and some additional details and applications, appear in [16].

Lemma 7.1. *Let $\{R_i, R_i'\}_{i\in\mathbb{Z}}$ be a bi-infinite sequence of l-dimensional windows contained in Λ. Assume that the following properties hold for all $i \in \mathbb{Z}$:*

(i) *$R_i \subseteq U^-$ and $R_i' \subseteq U^+$.*

(ii) R_i is correctly aligned with R'_{i+1} under the scattering map S.

(iii) for each pair R'_{i+1}, R_{i+1} and for each $L > 0$ there exists $L' > L$ such that R'_{i+1} is correctly aligned with R_{i+1} under the iterate $f^{L'}_{|\Lambda}$ of the restriction $f_{|\Lambda}$ of f to Λ.

Fix any bi-infinite sequence of positive real numbers $\{\varepsilon_i\}_{i \in \mathbb{Z}}$. Then there exist an orbit $(f^n(z))_{n \in \mathbb{Z}}$ of some point $z \in M$, an increasing sequence of integers $(n_i)_{i \in \mathbb{Z}}$, and some sequences of positive integers $\{N_i\}_{i \in \mathbb{Z}}, \{K_i\}_{i \in \mathbb{Z}}, \{M_i\}_{i \in \mathbb{Z}}$, such that, for all $i \in \mathbb{Z}$:

$$d(f^{n_i}(z), \Gamma) < \varepsilon_i,$$

$$d(f^{n_i + N_{i+1}}(z), f^{N_{i+1}}_{|\Lambda}(R'_{i+1})) < \varepsilon_{i+1},$$

$$d(f^{n_i - M_i}(z), f^{-M_i}_{|\Lambda}(R_i)) < \varepsilon_i,$$

$$n_{i+1} = n_i + N_{i+1} + K_{i+1} + M_{i+1}.$$

Proof. The idea of this proof is to 'thicken' some appropriate iterates of the windows R_i, R'_i in Λ to full dimensional windows W_i, W'_i in M, so that $\{W_i, W'_i\}_{i \in \mathbb{Z}}$ form a sequence of windows that are correctly aligned under some appropriate maps. We start with a brief sketch of the construction before we proceed to the formal proof. We constructs some copies $\bar{R}_i, \bar{R}'_{i+1}$ in Γ of R_i, R'_{i+1}, respectively, through the inverses of the wave maps (see Section 5.1). We then expand the rectangles $\bar{R}_i, \bar{R}'_{i+1}$ into the hyperbolic directions to produce a pair of windows $\bar{W}_i, \bar{W}'_{i+1}$, respectively, which are correctly aligned under the identity map. Then we take a backwards iterate of $\bar{W}_i$ such that $f^{-M_i}(\bar{W}_i)$ is sufficiently close to Λ, and we construct a new window W_i about $f^{-M_i}(R_i)$ such that W_i is correctly aligned with $\bar{W}_i$ under f^{M_i}. Similarly, we construct a window W'_{i+1} about $f^{N_i}(R'_{i+1})$ such that $\bar{W}'_{i+1}$ is correctly aligned with W'_{i+1} under $f^{N_{i+1}}$. We are given that we can align R'_{i+1} with R_{i+1} under some high enough iterate. Hence we can align W'_{i+1} with W_{i+1} under some iterate $f^{K_{i+1}}$. This construction can be continued inductively.

Notationwise, the N_i's are associated to forward iterations along the stable manifold, the M_i's to backwards iterations along the unstable manifold, and the K_i's to iterations following the inner dynamics of f restricted to Λ. See Fig. 8.

Step 1. Let $\{R, R'\}$ be a pair of l-dimensional windows of the type $\{R_i, R'_{i+1}\}$, and let $\{\varepsilon, \varepsilon'\}$ stand for the corresponding $\{\varepsilon_i, \varepsilon_{i+1}\}$. Let $\bar{R} = (\Omega^-_\Gamma)^{-1}(R)$ and $\bar{R}' = (\Omega^+_\Gamma)^{-1}(R')$ be the copies of R and R', respectively, in the homoclinic channel Γ. By making some arbitrarily small changes in the sizes of their exit and entry directions, we can alter the windows $\bar{R}$ and $\bar{R}'$ such that R is correctly aligned with $\bar{R}$ under $(\Omega^-_\Gamma)^{-1}$, $\bar{R}$ is correctly aligned with $\bar{R}'$ under the identity mapping, and $\bar{R}'$ is correctly aligned with R under Ω^+_Γ.

We 'thicken' the l-dimensional windows $\bar{R}$ and $\bar{R}'$ in Γ, which are correctly aligned under the identity mapping, to $(l + n_u + n_s)$-dimensional windows $\bar{W}$ and $\bar{W}'$, respectively, that are correctly aligned in M under the identity mapping as well. We now explain the 'thickening' procedure.

First, we describe how to thicken $\bar{R}$ to a full dimensional window $\bar{W}$. We choose some $0 < \bar{\delta} < \varepsilon$ and $0 < \bar{\eta} < \varepsilon$. At each point $x \in \bar{R}$ we choose an n_u-dimensional closed ball $\bar{B}_{\bar{\delta}}(x)$ of radius $\bar{\delta}$ centered at x and contained in $W^u(x^-)$, where $x^- = \Omega^-_\Gamma(x)$. Let $\bar{\Delta} := \bigcup_{x \in \bar{R}} \bar{B}^u_{\bar{\delta}}(x)$. Note that $\bar{\Delta}$ is contained in $W^u(\Lambda)$ and is homeomorphic to a $(l + n_u)$-dimensional rectangle. We define the exit set
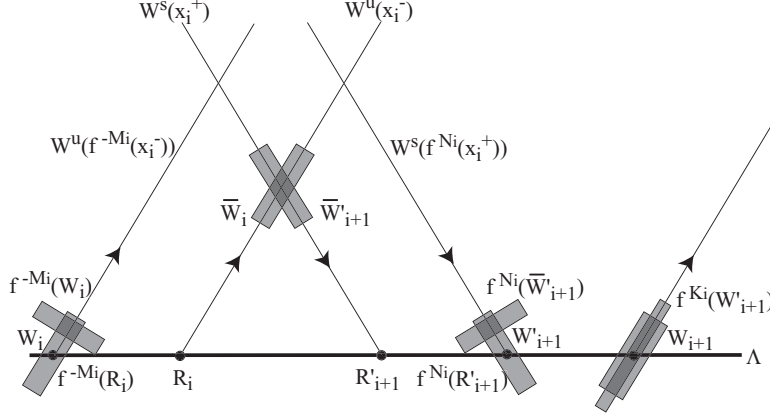
FIGURE 8. Schematic illustration of the construction of windows for the shadowing lemma. The windows R_i, R'_{i+1} are depicted as large dots.

and the entry set of this rectangle as follows:

$$(\bar{\Delta})^{\text{exit}} := \bigcup_{x \in (\bar{R})^{\text{exit}}} \bar{B}_{\bar{\delta}}^u(x) \cup \bigcup_{x \in \bar{R}} \partial \bar{B}_{\bar{\delta}}^u(x),$$

$$(\bar{\Delta})^{\text{entry}} := \bigcup_{x \in (\bar{R})^{\text{entry}}} \bar{B}_{\bar{\delta}}^u(x).$$

We consider the normal bundle N to $W^u(\Lambda)$. At each point $y \in \bar{\Delta}$, we choose an n_s-dimensional closed ball $\bar{B}_{\bar{\eta}}^s(y)$ centered at y and contained in the image of N_y under the exponential map $\exp_y : T_y M \to M$. We let $\bar{W} := \bigcup_{y \in \bar{\Delta}} \bar{B}_{\bar{\eta}}^s(y)$. By the Tubular Neighborhood Theorem (see, e.g., [7]), we have that for $\bar{\eta}$ sufficiently small, $\bar{W}$ is a homeomorphic copy of a $(l + n_u + n_s)$-dimensional rectangle. We now define the exit set and the entry set of $\bar{W}$ as follows:

$$(\bar{W})^{\text{exit}} := \bigcup_{y \in (\bar{\Delta})^{\text{exit}}} \bar{B}_{\bar{\eta}}^s(y),$$

$$(\bar{W})^{\text{entry}} := \bigcup_{y \in (\bar{\Delta})^{\text{entry}}} \bar{B}_{\bar{\eta}}^s(y) \cup \bigcup_{y \in (\bar{\Delta})} \partial \bar{B}_{\bar{\eta}}^s(y).$$

Second, we describe in a similar fashion how to thicken $\bar{R}'$ to a full dimensional window $\bar{W}'$. We choose $0 < \bar{\delta}' < \varepsilon'$ and $0 < \bar{\eta}' < \varepsilon'$. We consider the $(l + n_s)$-dimensional rectangle $\bar{\Delta}' := \bigcup_{x' \in \bar{R}'} \bar{B}_{\bar{\eta}'}^s(x') \subseteq W^s(\Lambda)$, where $\bar{B}_{\bar{\eta}'}^s(x')$ is the n-dimensional closed ball of radius $\bar{\eta}'$ centered at x' and contained in $W^s(x^+)$, with $x^+ = \Omega_\Gamma^+(x')$. Its exit set and entry sets are defined as follows:

$$(\bar{\Delta}')^{\text{exit}} := \bigcup_{x \in (\bar{R}')^{\text{exit}}} \bar{B}_{\bar{\eta}'}^s(x'),$$

$$(\bar{\Delta}')^{\text{entry}} := \bigcup_{x' \in (\bar{R}')^{\text{entry}}} \bar{B}_{\bar{\eta}'}^s(x') \cup \bigcup_{x \in (\bar{R}')} \partial \bar{B}_{\bar{\eta}'}^s(x').$$

We define $\bar{W}' := \bigcup_{y' \in \bar{\Delta}'} \bar{B}^u_{\bar{\delta}'}(y')$, where $\bar{B}^u_{\bar{\delta}'}(y')$ is the n-dimensional closed ball centered at y' and contained in the image of $N'_{y'}$ under the exponential map $\exp_{y'} : T_{y'}M \to M$, with N' being the normal bundle to $W^s(\Lambda)$. For $\bar{\delta}' > 0$ sufficiently small $\bar{W}'$ is a homeomorphic copy of a $(l + n_u + n_s)$-dimensional rectangle. The exit set and the entry set of $\bar{W}'$ are defined by:

$$(\bar{W}')^{\text{exit}} := \bigcup_{y' \in (\bar{\Delta}')^{\text{exit}}} \bar{B}^u_{\bar{\delta}'}(y') \cup \bigcup_{y' \in (\bar{\Delta}')} \partial \bar{B}^u_{\bar{\delta}'}(y'),$$

$$(W')^{\text{entry}} := \bigcup_{y' \in (\bar{\Delta}')^{\text{entry}}} \bar{B}^u_{\bar{\delta}'_{i+1}}(y').$$

This completes the description of the thickening of the l-dimensional window $\bar{R}$ into a $(l + n_u + n_s)$-dimensional window $\bar{W}$, and of the thickening of the l-dimensional window $\bar{R}'$ into a $(l + n_u + n_s)$-dimensional window $\bar{W}'$. Note that by construction $\bar{W}$ is contained in an ε-neighborhood of Λ and $\bar{W}'$ is contained in an ε'-neighborhood of Γ.

In order to make $\bar{W}$ correctly aligned with $\bar{W}'$ under the identity map, we choose $\bar{\delta}'$ sufficiently small relative to $\bar{\delta}$, and $\bar{\eta}$ sufficiently small relative to $\bar{\eta}'$.

Step 2. We take a negative iterate $f^{-M}(\bar{R})$ of $\bar{R}$, where $M > 0$. We have that $f^{-M}(\Gamma)$ is ε-close to Λ in the C^1-topology, for all M sufficiently large. The vectors tangent to the fibers $W^u(x^-)$ in $\bar{R}$ are contracted, and the vectors transverse to $W^u(\Lambda)$ along $\bar{R}$ are expanded by the derivative of f^{-M}. We choose M sufficiently large so that $f^{-M}(\bar{R})$ is ε-close to $f^{-M}(R)$.

We now construct a window W about $f^{-M}(R)$ that is correctly aligned with $f^{-M}(\bar{W})$ under the identity. Note that each closed ball $\bar{B}^u_\delta(x)$, which is a part of $\bar{\Delta}$, gets exponentially contracted as it is mapped onto $W^u(f^{-M}(x^-))$ by f^{-M}. By the Lambda Lemma (see the version in [44]), each closed ball $\bar{B}^s_\eta(y)$, $y \in \bar{\Delta}$, C^1-approaches some subset of $W^s(f^{-M}(y^-))$ under f^{-M}, as $M \to \infty$. For M sufficiently large, we can assume that $f^{-M}(\bar{B}^s_{\bar{\eta}}(y))$ is ε-close to some ball in $W^s(f^{-M}(y^-))$ in the C^1-topology, for all $y \in \bar{\Delta}$. We fix M with the above properties. As R is correctly aligned with $\bar{R}$ under $(\Omega^-_\Gamma)^{-1}$, we have that $f^{-M}(R)$ is correctly aligned with $f^{-M}(\bar{R})$ under $(\Omega^-_{f^{-M}(\Gamma)})^{-1}$. In other words, $f^{-M}(R)$ is correctly aligned under the identity mapping with the projection of $f^{-M}(\bar{R})$ onto Λ along the unstable fibers.

To define the window W, we use a local linearization of the normally hyperbolic invariant manifold. By Theorem 1 in [49], there exists a homeomorphism h of an open neighborhood of $(E^u \oplus E^s)_{|\Lambda}$ to an open neighborhood of Λ in M such that $h \circ Df = f \circ h$. We select a point $x \in f^{-M}(R)$. Since R is contractible the bundles are trivial on $f^{-M}(R)$ and we can identify $(E^u \oplus E^s)_{|f^{-M}(R)}$ with $f^{-M}(R) \times E^u_x \times E^s_x$. Let us consider $0 < \delta < \varepsilon$ and $0 < \eta < \varepsilon$. We define a window W as

$$W = h(f^{-M}(R) \times \bar{B}^u_\delta(0) \times \bar{B}^s_\eta(0)),$$

where $\bar{B}^u_\delta(0)$ is the closed ball centered at 0 of radius δ in E^u_x and $\bar{B}^s_\eta(0)$ is the closed ball centered at 0 of radius η in E^s_x. We define the exit set of W as

$$(W)^{\text{exit}} = h(f^{-M}(\bar{R}) \times \partial \bar{B}^u_\delta(0) \times \bar{B}^s_\eta(0)) \cup h(f^{-M}(\bar{R}^{\text{exit}}) \times B^u_\delta(0) \times \bar{B}^s_\eta(0)).$$

Similarly, the entry set of W is defined as

$$(W)^{\text{entry}} = h(f^{-M}(\bar{R}) \times \bar{B}^u_\delta(0) \times \partial \bar{B}^s_\eta(0)) \cup h(f^{-M}(\bar{R}^{\text{entry}}) \times B^u_\delta(0) \times \bar{B}^s_\eta(0)).$$

In order to ensure the correct alignment of W with $f^{-M}(\bar{W})$ under the identity map, we choose δ, η such that $h(f^{-M}(R) \times \bar{B}^u_\delta(0) \times \{0\})$ is correctly aligned with $f^{-M}(\bar{\Delta})$ under the identity map (the exit sets of both windows being in the unstable directions), and that each closed ball $f^{-M}(\bar{B}^s_\eta)$ intersects W in a closed ball that is contained in the interior of $f^{-M}(\bar{B}^s_\eta)$. The existence of suitable δ, η follows from the exponential contraction of $\bar{\Delta}$ under negative iteration, and from the Lambda Lemma applied to $\bar{B}^s_\eta(y)$ under negative iteration.

In a similar fashion, we construct a window W' contained in an ε'-neighborhood of Λ such that $\bar{W}'$ is correctly aligned with W' under f^N. The window W', and its entry and exit sets, are defined by:

$$
\begin{aligned}
W' =& h(f^N(R') \times \bar{B}^u_{\delta'}(0) \times \bar{B}^s_{\eta'}(0)), \\
(W')^{\text{exit}} =& h(f^N(R') \times \partial\bar{B}^u_{\delta'}(0) \times \bar{B}^s_{\eta'}(0)) \\
& \cup h(f^N((R')^{\text{exit}}) \times \bar{B}^u_{\delta'}(0) \times \bar{B}^s_{\eta'}(0)), \\
(W')^{\text{entry}} =& h(f^N(R') \times \bar{B}^u_{\delta'}(0) \times \partial\bar{B}^s_{\eta'}(0)) \\
& \cup h(f^N((R')^{\text{entry}}) \times \bar{B}^u_{\delta'}(0) \times \bar{B}^s_{\eta'}(0)),
\end{aligned}
$$

for some appropriate choices of radii $0 < \delta', \eta' < \varepsilon'$.

Step 3. Suppose that we have constructed, as in Step 2, a window W' about the l-dimensional rectangle $f^N(R') \subseteq \Lambda$, and a window W about the l-dimensional rectangle $f^{-M}(R) \subseteq \Lambda$. Under positive iterations, the rectangle $\bar{B}^u_{\delta'}(0) \times \bar{B}^s_{\eta'}(0) \subseteq E^u \oplus E^s$ gets exponentially expanded in the unstable direction and exponentially contracted in the stable direction by Df. Thus $\bar{B}^u_{\delta'}(0) \times \bar{B}^s_{\eta'}(0)$ gets correctly aligned with $\bar{B}^u_\delta(0) \times \bar{B}^s_\eta(0)$ under some power Df^L of Df, provided L is sufficiently large. This implies that $f^L(h(\{x\} \times \bar{B}^u_{\delta'}(0) \times \bar{B}^s_{\eta'}(0)))$ is correctly aligned with $h(f^L(x) \times \bar{B}^u_\delta(0) \times \bar{B}^s_\eta(0))$ under the identity map (both rectangles are contained in $h(f^L(x) \times E^u \times E^s)$).

By assumption (iii), there exists $L' > \max\{L, N+M\}$ such that R' is correctly aligned with R under $f^{L'}$. This means that $f^N(R')$ is correctly aligned with $f^{-M}(R)$ under f^K with $K := L' - N - M > 0$.

The product property of correctly aligned windows implies that W' is correctly aligned with W under f^K, provided that K is chosen as above.

Step 4. We will now describe the process of constructing, based on Steps 1, 2, and 3, two bi-infinite sequences of windows $\{W_i, W'_i\}_{i\in\mathbb{Z}}$ and $\{\bar{W}_i, \bar{W}'_i\}_{i\in\mathbb{Z}}$ such that, for all $i \in \mathbb{Z}$, W_i is correctly aligned with $\bar{W}_i$ under f^{N_i}, $\bar{W}_i$ is correctly aligned with $\bar{W}'_{i+1}$ under the identity mapping, $\bar{W}'_{i+1}$ is correctly aligned with W'_{i+1} under f^{M_i}, and W'_{i+1} is correctly aligned with W_{i+1} under $f^{K_{i+1}}$. The point of this step is that we can repeatedly choose the rectangles and the parameters at Steps 1, 2, and 3, in a consistent way, in order to produce infinite sequences of correctly aligned windows.

Staring with $i = 0$ and continuing for all $i \geq 0$, we do the following.

For a given pair of l-dimensional windows $\{R_i, R'_{i+1}\}$, we consider the corresponding copies $\{\bar{R}_i, \bar{R}'_{i+1}\}$ in Γ such that $\bar{R}_i$ is correctly aligned with $\bar{R}'_{i+1}$ under the identity. As in Step 1, we construct a pair of $(l + n_u + n_s)$-dimensional windows $\bar{W}_i$ about $\bar{R}_i$, and $\bar{W}'_{i+1}$ about $\bar{R}'_{i+1}$. The size of $\bar{W}_i$ in the hyperbolic directions is given by some disks radii $\bar{\delta}_i, \bar{\eta}_i < \varepsilon_i$, and that of $\bar{W}'_{i+1}$ by some disk

radii $\bar\delta'_{i+1}, \bar\eta'_{i+1} < \varepsilon_{i+1}$. We choose the quantities $\bar\delta_i, \bar\eta_i, \bar\delta'_{i+1}, \bar\eta'_{i+1}$ such that $\bar W_i$ is correctly aligned with $\bar W'_{i+1}$ under the identity.

Then, we consider the pair of l-dimensional windows $\{R_{i+1}, R'_{i+2}\}$ in Λ, and their corresponding copies $\{\bar R_{i+1}, \bar R'_{i+2}\}$ in Γ. As in Step 1, we construct a pair of $(l + n_u + n_s)$-dimensional windows $\bar W_{i+1}$ about $\bar R_{i+1}$, and $\bar W'_{i+2}$ about $\bar R'_{i+2}$. By choosing the quantities $\bar\delta_{i+1}, \bar\eta_{i+1}, \bar\delta'_{i+2}, \bar\eta'_{i+2}$ as in Step 1 we can ensure that $\bar W_{i+1}$ is correctly aligned with $\bar W'_{i+2}$.

We choose N_{i+1}, M_{i+1} large enough so that $f^{N_{i+1}}(\bar W'_{i+1})$ is contained in an ε_{i+1}-neighborhood of $f^{N_{i+1}}(R'_{i+1})$, and $f^{-M_{i+1}}(\bar W_{i+1})$ is contained in an ε_{i+1}-neighborhood of $f^{-M_{i+1}}(R_{i+1})$. As in Step 2, we construct a window W'_{i+1} about $f^{N_1}(R'_{i+1})$ such that $\bar W'_{i+1}$ correctly aligned with W'_{i+1} under $f^{N_{i+1}}$, and a window W_{i+1} about $f^{-M_1}(R_{i+1})$ such that W_{i+1} correctly aligned with $\bar W_{i+1}$ under $f^{M_{i+1}}$. This amounts to choosing the quantities $\delta'_{i+1}, \eta'_{i+1}, \delta_{i+1}, \eta_{i+1}$ as in Step 2 in order to ensure the correct alignment of the windows.

Then, we choose K_{i+1} sufficiently large, and at least as large as $N_{i+1} + M_{i+1}$, such that W'_{i+1} is correctly aligned with W_{i+1} under $f^{K_{i+1}}$. At this point, we have that $\bar W'_{i+1}$ is correctly aligned with W'_{i+1} under $f^{N_{i+1}}$, W'_{i+1} is correctly aligned with W_{i+1} under $f^{K_{i+1}}$, and W_{i+1} is correctly aligned with $\bar W_{i+1}$ under $f^{M_{i+1}}$.

Inductively, we obtain two sequences of windows $\{W_i, W'_i\}_{i\geq 0}$ and $\{\bar W_i, \bar W'_i\}_{i\geq 0}$ that satisfy the desired correct alignment conditions for all $i \geq 0$. A similar inductive construction of windows can be done backwards starting with W_0, W'_1.

In the end, we obtain two bi-infinite sequences of windows $\{W_i, W'_i\}_{i\in\mathbb{Z}}$ and $\{\bar W_i, \bar W'_i\}_{i\geq 0}$ that satisfy the desired correct alignment conditions for all $i \in \mathbb{Z}$. By Theorem 5.6, there exists an orbit $\{f^n(z)\}_{n\in\mathbb{Z}}$ with $f^{n_i}(z) \in \bar W_i \cap \bar W'_{i+1}$, $f^{n_i + N_{i+1}}(z) \in W'_{i+1}$, $f^{n_i + N_{i+1} + K_{i+1}}(z) \in W_{i+1}$, $f^{n_i + N_{i+1} + K_{i+1} + M_{i+1}}(z) \in \bar W_{i+1} \cap \bar W'_{i+2}$, for all $i \in \mathbb{Z}$. Thus $n_{i+1} = n_i + N_{i+1} + K_{i+1} + M_{i+1}$. Such an orbit $\{f^n(z)\}_{n\in\mathbb{Z}}$ satisfies the properties required by Lemma 7.1. $\square$

8. Construction of correctly aligned windows

In this section we prove Theorem 2.1. The methodology consists of constructing 2-dimensional windows in Λ about the prescribed invariant primary tori, BZI's, and Aubry-Mather sets inside the BZI's. The successive pairs of windows are correctly aligned under the scattering map alternatively with powers of the inner map. Lemma 7.1 implies that there exist trajectories that follow these windows.

8.1. **Construction of correctly aligned windows across a BZI.** In this section we will construct correctly aligned windows across a BZI between two successive transition chains of tori. On each side of the BZI we will choose a one-sided neighborhood of a point on the boundary torus, and we will use Theorem 6.1 or Theorem 6.2 to cross over the BZI. These one-sided neighborhoods are of a special type: their boundaries are images of some transition tori under the inner or outer dynamics. We will construct some windows about the boundaries of these one-sided neighborhoods. Then we will construct some other windows about the corresponding transition tori; these windows will be chosen so that they have a pair of sides lying on some nearby tori. We will use this feature later to connect sequence of windows across the BZI's with sequences of windows along the transition chains.

Consider an annular region Λ_k in Λ that is a BZI, and is between two transition chains of invariant tori, as in (A5). To simplify notation, we denote the tori at the

boundary of Λ_k by T_a and T_b. We choose a pair of transition tori T_i, T_j in Λ as in (A5), ordered as follows: $T_j \prec T_i \prec T_a$. These tori are outside of the BZI Λ_k and on the same side of it as T_a. By (A5-iv) there exist $T_{i'} \prec T_i$ and $T_{j'} \prec T_j$ such that $T_{i'}$ is ε_i-close to T_i and $T_{j'}$ is ε_j-close to T_j, in the C^0 topology. By (A5-ii) $S(T_j)$ intersects T_i in a topologically transverse manner, so both $S(T_j)$ and $S(T_{j'})$ intersect both T_i and $T_{i'}$ in a topologically transverse manner, provided $T_{i'}, T_{j'}$ are sufficiently close to T_i, T_j, respectively.

Since S is a diffeomorphism, $T_i, T_{i'}$ and the images of $T_j, T_{j'}$ under S form a topological rectangle $D_{iji'j'}$ in Λ. This rectangle may not be contained in the domain U^- of the scattering map S. Provided that we choose the tori $T_i, T_{i'}$ sufficiently C^0-close to one another, and also $T_j, T_{j'}$ sufficiently C^0-close to one another, the rectangle $D_{iji'j'}$ will be sufficiently small so that some iterate $f_{|\Lambda}^{K_a}$ of $f_{|\Lambda}$ takes the rectangle $D_{iji'j'}$ into a rectangle $f_{|\Lambda}^{K_a}(D_{iji'j'})$ inside U^-. This is possible since each torus intersects U^- by (A5-i), and the motion on the tori is topologically transitive by (A5-iii). The rectangle $f_{|\Lambda}^{K_a}(D_{iji'j'})$ has a pair of sides lying on the tori $T_i, T_{i'}$, and the other pair of sides on $(f_{|\Lambda}^{K_a} \circ S)(T_j), (f_{|\Lambda}^{K_a} \circ S)(T_{j'})$. The curves $(f_{|\Lambda}^{K_a} \circ S)(T_j), (f_{|\Lambda}^{K_a} \circ S)(T_{j'})$ are topologically transverse to both $T_i, T_{i'}$. By assumption (A5-ii), $S(T_i)$ topologically crosses T_a, so $S^{-1}(T_a)$ topologically crosses T_i. We can ensure that the interior of $f_{|\Lambda}^{K_a}(D_{iji'j'})$ intersects $S^{-1}(T_a)$ by choosing K_a sufficiently large, and the tori in each pair $T_i, T_{i'}$ and $T_j, T_{j'}$ sufficiently C^0-close to one another. This implies that the image of $f_{|\Lambda}^{K_a}(D_{iji'j'})$ under S is a topological rectangle in Λ which intersects T_a, and the intersection of $S(f_{|\Lambda}^{K_a}(D_{iji'j'}))$ with Λ_k forms a one-sided neighborhood in Λ_k of some part of T_a. The boundary of $S(f_{|\Lambda}^{K_a}(D_{iji'j'})) \cap \Lambda_k$ consists of arcs of the curves $T_a, S(T_i), S(T_{i'}), S \circ f_{|\Lambda}^{K_a} \circ S(T_j), S \circ f_{|\Lambda}^{K_a} \circ S(T_{j'})$.

We make a similar construction on the other side of the BZI Λ_k. We choose a pair of transition tori T_k, T_l with $T_b \prec T_k \prec T_l$, outside of the BZI Λ_k and on the same side as T_b. We also choose $T_k \prec T_{k'}$ and $T_l \prec T_{l'}$ such that $T_{k'}$ is ε_k-close to T_k and $T_{l'}$ is ε_l-close to T_l, and $S^{-1}(T_l), S^{-1}(T_{l'})$ are topologically transverse to both $T_k, T_{k'}$, and so they form a a topological rectangle $D_{klk'l'}$. There exists K_b sufficiently large such that $f_{|\Lambda}^{-K_b}(D_{klk'l'}) \subseteq U^-$, its interior intersects $S(T_b)$, and $S^{-1}(f_{|\Lambda}^{-K_b}(D_{klk'l'}))$ forms a one-sided neighborhood in Λ_k of some part of T_b. The boundary of $S^{-1}(f_{|\Lambda}^{K_b}(D_{klk'l'})) \cap \Lambda_k$ consists of arcs of the curves $T_b, S(T_k), S(T_{k'}), S^{-1} \circ f_{|\Lambda}^{K_b} \circ S^{-1}(T_l), S^{-1} \circ f_{|\Lambda}^{K_b} \circ S^{-1}(T_{l'})$.

At this stage we have obtained in Λ_k a one-sided neighborhood $(S \circ f_{|\Lambda}^{K_a})(D_{iji'j'})$ of an arc in T_a, and a one-sided neighborhood $(S^{-1} \circ f_{|\Lambda}^{-K_b})(D_{klk'l'})$ of an arc in T_b.

If we are under the assumptions (A1)-(A6), Theorem 6.1 yields a point $x_a \in \partial(S \circ f_{|\Lambda}^{K_a})(D_{iji'j'})$ whose image $x_b = f_{|\Lambda}^{K_{ab}}(x_a)$ under some power $f_{|\Lambda}^{K_{ab}}$ lies on $\partial(S^{-1} \circ f_{|\Lambda}^{-K_b})(D_{klk'l'})$.

If we also assume (A7), Theorem 6.2 yields a point $x_a \in \partial(S \circ f_{|\Lambda}^{K_a})(D_{iji'j'})$ with $x_b = f_{|\Lambda}^{K_{ab}}(x_a) \in \partial(S^{-1} \circ f_{|\Lambda}^{-K_b})(D_{klk'l'})$ as above, satisfying the additional
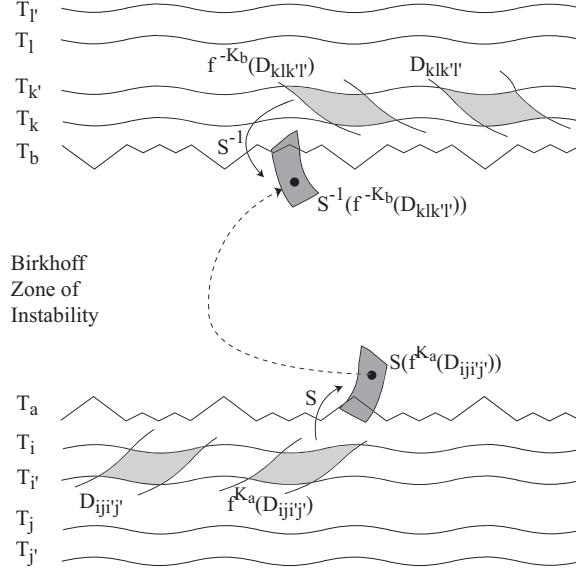
FIGURE 9. Orbits across a BZI

conditions

$$\pi_\phi(f^j(w_s^k)) < \pi_\phi(f^j(x_a)) < \pi_\phi(f^j(\bar{w}_s^k)),$$

for each $s \in \{1, \dots, s_k\}$, where $w_s^k, \bar{w}_s^k \in \Sigma_{\omega_s^k}$, and for all j within a certain interval of integers. The trajectories of all points sufficiently close to x_a will satisfy these conditions as well.

In either case, there exist an arc $\bar{e}_a' \subseteq \partial(S \circ f_{|\Lambda}^{K_a})(D_{iji'j'})$ containing x_a, and another arc $\bar{e}_b \subseteq \partial(S^{-1} \circ f_{|\Lambda}^{-K_b})(D_{klk'l'})$ containing x_b, such that $f_{|\Lambda}^{K_{ab}}(\bar{e}_a')$ is topologically transverse to $\bar{e}_b$ at x_b.

The arc $\bar{e}_a'$ lies on one of the sets $S(T_i), S(T_{i'}), (S \circ f_{|\Lambda}^{K_a} \circ S)(T_j), (S \circ f_{|\Lambda}^{K_a} \circ S)(T_{j'})$. Similarly, the arc $\bar{e}_b$ lies on one of the sets $S^{-1}(T_k), S^{-1}(T_{k'}), (S^{-1} \circ f_{|\Lambda}^{-K_b} \circ S^{-1})(T_l), (S^{-1} \circ f_{|\Lambda}^{-K_b} \circ S^{-1})(T_{l'})$.

We define a 2-dimensional window R_a' about $\bar{e}_a'$, and a 2-dimensional window R_b about $\bar{e}_b$ such that R_a' is correctly aligned with R_b under $f_{|\Lambda}^{K_{ab}}$. Informally, the exit direction of R_a' is along $\bar{e}_a'$, and the exit direction of R_b is across $\bar{e}_b$. The formal construction now follows. Since the arc $\bar{e}_a'$ is an embedded 1-dimensional C^0-submanifold of Λ, there exists a C^0-local parametrization $\chi_a' : \mathbb{R}^2 \to \Lambda$ such that $\chi_a'([0,1] \times \{0\}) = \bar{e}_a'$, provided $\bar{e}_a'$ is sufficiently small. Then

$$R_a' = \chi_a([0,1] \times [-\eta_a', \eta_a']),$$

is a topological rectangle. We define the exit set of R_a' as

$$R_a'^{\text{exit}} = \chi_a(\partial[0,1] \times [-\eta_a', \eta_a']).$$

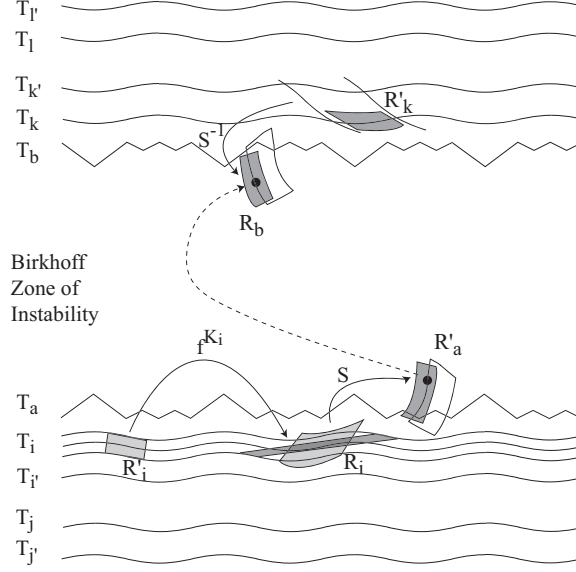The definition of the entry set of of R_a' follows by default.

FIGURE 10. Construction of windows near the boundaries of a BZI
– case 1

Similarly, there exists a C^0-local parametrization $\chi_b : \mathbb{R}^2 \to \Lambda$ such that $\chi_b(\{0\} \times [0,1]) = \bar{e}_b$, and
$$R_b = \chi_b([-\delta_b, \delta_b] \times [0,1]),$$
is a topological rectangle. We define the exit set of R_b as
$$R_b^{\text{exit}} = \chi_b(\partial[-\delta_b, \delta_b] \times [0,1]).$$

By choosing η'_a, δ_b sufficiently small, we ensure that R'_a is correctly aligned with R_b under $f^{K_{ab}}_{|\Lambda}$ (see Definition 5.5). See Figure 10.

We now construct other windows outside the BZI Λ_k. We consider two cases: first case, when the arc $\bar{e}'_a$ is a part of $S(T_i)$ or $S(T_{i'})$; second case, when $\bar{e}'_a$ is a part of $(S \circ f^{K_a}_{|\Lambda} \circ S)(T_j)$ or $(S \circ f^{K_a}_{|\Lambda} \circ S)(T_{j'})$.

Case 1. In the first case, when $\bar{e}'_a$ is a part of $S(T_i)$ or $S(T_{i'})$ we proceed with the construction as follows. Let us say that $\bar{e}'_a$ is a part of $S(T_i)$. We take the inverse image of R'_a by S. This is a topological rectangle $S^{-1}(R'_a)$ about the torus T_i. We construct a window R_i about an arc $\bar{e}_i$ in T_i, such that R_i is correctly aligned with $S^{-1}(R'_a)$ under the identity map, and with the exit direction of R_i in the direction of T_i. Let $\bar{e}_i$ be an arc in T_i, and $\chi_i : \mathbb{R}^2 \to \Lambda$ be a local parametrization with $\chi_i([0,1] \times \{0\}) = \bar{e}_i$. We define
$$R_i = \chi_i([0,1] \times [-\eta_i, \eta_i])$$
$$R_i^{\text{exit}} = \chi_i(\partial[0,1] \times [-\eta_i, \eta_i]),$$

where $\eta_i > 0$ is sufficiently small. By choosing the arc $\bar{e}_i$ sufficiently large so that $\bar{e}_i \supseteq S^{-1}(\bar{e}'_a)$, and by choosing $\eta_i > 0$ sufficiently small, we can ensure that R_i is correctly aligned with $S^{-1}(R'_a)$ under the identity map, or, equivalently, R_i is correctly aligned with R'_a under S. See Figure 10.

We construct another window R_i' about T_i such that R_i' is correctly aligned with R_i under some power $f_{|\Lambda}^{K_i}$ of $f_{|\Lambda}$, with $K_i > 0$. The window R_i' will have its exit direction across the torus T_i. We choose a pair of invariant primary tori $T_i^{\text{lower}} \prec T_i \prec T_i^{\text{upper}}$, with $T_i^{\text{lower}}, T_i^{\text{upper}}$ located ε_i-close to T_i in the C^0-topology. Such tori exist due to the assumption that T_i is not an end torus in the transition chain, as in (A5-iv). Moreover, we choose the tori $T_i^{\text{lower}}, T_i^{\text{upper}}$ sufficiently close to T_i so that the components of R_i^{entry} are outside the annulus between T_i^{lower} and T_i^{upper}, with one component on one side and the other component on the other side of this annulus. We choose an arc $\bar{e}_i' \subseteq T_i$ that lies on an edge of the topological rectangle $D_{iji'j'}$. Let $\chi_i' : \mathbb{R}^2 \to \Lambda$ be a C^0-local parametrization with $\chi_i'(\{0\} \times [0,1]) = \bar{e}_i'$, such that for some δ_i' sufficiently small, $\chi_i'(\{-\delta_i'\} \times [0,1]) \subseteq T_i^{\text{lower}}$ and $\chi_i'(\{\delta_i'\} \times [0,1]) \subseteq T_i^{\text{upper}}$. We define R_i' by:

$$R_i' = \chi_i'([-\delta_i', \delta_i'] \times [0,1]),$$
$$R_i'^{\text{exit}} = \chi_i'(\partial[-\delta_i', \delta_i'] \times [0,1]).$$

The exit set components of R_i' lie on the tori $T_i^{\text{upper}}, T_i^{\text{lower}}$ neighboring T_i.

Since the motion on the tori is topologically transitive, by (A5-iii), there exists $K_i > 0$ such that R_i' is correctly aligned with R_i under $f_{|\Lambda}^{K_i}$. Indeed, to achieve correct alignment of these windows in the covering space of the annulus we only have to choose K_i sufficiently large so that the two components of $R_i'^{\text{exit}}$, which are two arcs in T_i^{lower} and T_i^{upper}, are mapped by $f_{|\Lambda}^{K_i}$ on the opposite sides of the part of R_i between T_i^{lower} and T_i^{upper}. We note that the number of iterates K_i of $f_{|\Lambda}$ needed to make R_i' correctly aligned with R_i may be different than the number of iterates K_a which takes the topological rectangle $D_{iji'j'}$ onto $f_{|\Lambda}^{K_a}(D_{ihi'j'})$. See Figure 10. The conclusion of this step is that we obtain the window R_i' around T_i, with its exit direction across T_i, such that R_i' is correctly aligned with R_i under $f_{|\Lambda}^{K_i}$. Both windows R_i', R_i are contained in an ε_i-neighborhood of T_i.

In the case when the edge $\bar{e}_a'$ of $D_{iji'j'}$ is a part of $S(T_{i'})$ instead of $S(T_i)$, the construction goes similarly to the one above.

Case 2. We now consider the second case, when the arc $\bar{e}_a'$ of $\partial(S \circ f_{|\Lambda}^{K_a})(D_{iji'j'})$ is a part of $(S \circ f_{|\Lambda}^{K_a} \circ S)(T_j)$ or $(S \circ f_{|\Lambda}^{K_a} \circ S)(T_{j'})$. Let us say that $\bar{e}_a'$ is a part of $(S \circ f_{|\Lambda}^{K_a} \circ S)(T_j)$. We construct a window R_a' in Λ about $\bar{e}_a'$ as before; the exit set of R_a' is in a direction along $\bar{e}_a'$, and the size of R_a' in the direction across R_a is given by some parameter η_a'. See Figure 11. We consider the inverse image $S^{-1}(R_a')$ of R_a' by S, which is a topological rectangle about $(f_{|\Lambda}^{K_a} \circ S)(T_j)$. Let $\bar{e}_j''$ be an arc in $(f_{|\Lambda}^{K_a} \circ S)(T_j)$ such that $S(\bar{e}_j'') \supset \bar{e}_a'$, and let $\chi_j'' : \mathbb{R}^2 \to \Lambda$ be a local parametrization with $\chi_j''([0,1] \times \{0\}) = \bar{e}_j''$. We define R_j'' by:

$$R_j'' = \chi_j''([0,1] \times [-\eta_j'', \eta_j'']),$$
$$R_j''^{\text{exit}} = \chi_j''(\partial[0,1] \times [-\eta_j'', \eta_j'']).$$

By choosing $\eta_j'' > 0$ sufficiently small, we can ensure that R_j'' is correctly aligned with aligned with R_a' under S. The window R_j'' is in a neighborhood of an edge of the topological rectangle $f_{|\Lambda}^{K_a}(D_{iji'j'})$. The exit set of R_j'' is in the direction of $(f_{|\Lambda}^{K_a} \circ S)(T_j)$. See Figure 11.
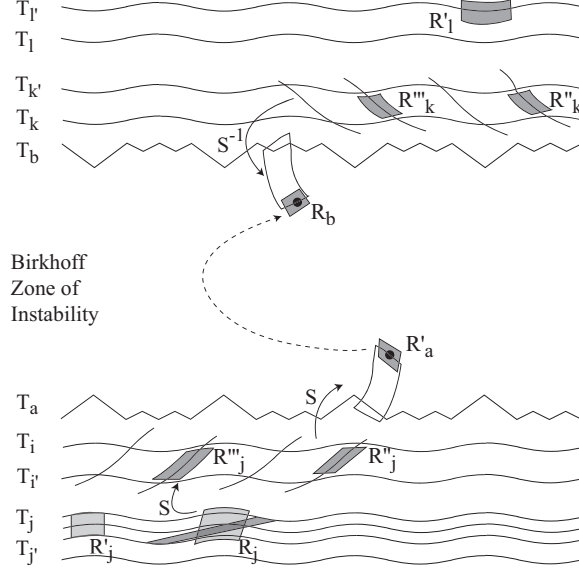
FIGURE 11. Construction of windows near the boundaries of a BZI
– case 2

We construct a window R_j''' about $S(T_j)$ such that R_j''' is correctly aligned with R_j'' under $f_{|\Lambda}^{K_a}$. We define

$$R_j''' = \chi_j'''([0,1] \times [-\eta_j''', \eta_j''']),$$
$$R_j'''^{\text{exit}} = \chi_j'''(\partial[0,1] \times [-\eta_j''', \eta_j''']),$$

where $\bar{e}_j'''$ is an arc of $S(T_j)$ with $f_{|\Lambda}^{K_a}(\bar{e}_j''') \supseteq \bar{e}_j''$, $\chi_j''' : \mathbb{R}^2 \to \Lambda$ is a local parametrization with $\chi_j'''([0,1] \times \{0\}) = \bar{e}_j'''$, and $\eta_j''' > 0$ is sufficiently small. The arc $\bar{e}_j'''$ is contained in one of the edges of the topological rectangle $D_{iji'j'}$. For suitable $\eta_j''' > 0$, we can ensure that R_j''' is correctly aligned with aligned with R_j'' under $f_{|\Lambda}^{K_a}$. Moreover, we choose R_j''', R_j'' so that these windows are both contained in an ε_j neighborhood of T_j. The exit set of R_j''' is in a direction along $S(T_j)$.

We take the inverse image $S^{-1}(R_j''')$ of R_j''' under S, which is a topological rectangle about an arc in T_j. In a fashion similar to Case 1, we construct two windows R_j', R_j about T_j such that R_j' is correctly aligned with R_j under some power $f_{|\Lambda}^{K_j}$ of $f_{|\Lambda}$, and R_j is correctly aligned with R_j''' under S. The exit of R_j' is chosen in a direction across T_j, and the exit set components $R_j'^{\text{exit}}$ of R_j' lie on two invariant primary tori $T_j^{\text{upper}}, T_j^{\text{lower}}$ neighboring T_j, that are ε_j-close to T_j and contain T_j between them. The exit of R_j is in a direction along T_j. The windows R_j', R_j are chosen to lie in an ε_j neighborhood of T_j relative to the C^0-topology.

The case when the arc $\bar{e}_a'$ is a part of $(S \circ f_{|\Lambda}^{K_a} \circ S)(T_{j'})$ is treated similarly.

This concludes the construction of correctly aligned windows in Λ, starting with the window R_a' about T_a, and moving backwards along transition tori that are on the same side as T_a of the BZI Λ_k.

This construction yields a sequence of windows of the type

$$(8.1) \qquad\qquad R_i', R_i, R_a',$$

in the first case, or

$$(8.2) \qquad\qquad R_j', R_j, R_j''', R_j'', R_a',$$

in the second case. In the first case, R_i' is correctly aligned with R_i under some power $f_{|\Lambda}^{K_i}$ of the inner map, and R_i is correctly aligned with R_a' under the outer map S. The exit direction of R_i' is across the torus T_i, and its exit set components lie on some invariant primary tori that are ε_i-close to T_i. The windows R_i', R_i are contained in an ε_i neighborhood of T_i. In the second case, R_j' is correctly aligned with R_j under some power $f_{|\Lambda}^{K_j}$ of the inner map, R_j is correctly aligned with R_j''' under the outer map S, R_j''' is correctly aligned with R_j'' under some power $f_{|\Lambda}^{K_a}$ of the inner map, and R_j'' is correctly aligned with R_a' under the outer map S. The exit direction of R_j is across the torus T_j, and its exit set components lie on some invariant primary tori that are ε_j-close to T_j. The windows R_j', R_j are contained in an ε_j neighborhood of T_j, and the windows R_j''', R_j'' are contained in an ε_i neighborhood of T_i.

We proceed with a similar construction on the other side of the BZI between Λ_k, that is, on the same side of the BZI as T_b. We have already defined the window R_b about T_b that R_a' is correctly aligned with R_b under $f_{|\Lambda}^{K_{ab}}$. Starting with the window R_b and moving forward along the transition chain T_b, T_k, T_l, we construct a sequence of windows of the type

$$(8.3) \qquad\qquad R_b, R_k'$$

or of the type

$$(8.4) \qquad\qquad R_b, R_k''', R_k'', R_l',$$

satisfying the correct alignment conditions below. In the first case, R_b is correctly aligned with R_k' under the outer map S. The exit direction of R_k' is across T_k, and the exit set components lie on two invariant primary tori ε_k-close to T_k. Moreover, R_k' is contained in an ε_k-neighborhood of T_k. In the second case, R_b' is correctly aligned with R_k''' under the outer map S, R_k''' is correctly aligned with R_k'' under some power $f_{|\Lambda}^{K_b}$ of the inner map, and R_k'' is correctly aligned with R_l' under the outer map S. The exit direction of R_l' is across T_l, and its exit set components lie on some invariant primary tori that are ε_l-close to T_l. The windows R_k''', R_k'' are contained in an ε_k-neighborhood of T_k, and R_l' is contained in an ε_l-neighborhood of T_l.

Similar statements apply when instead of windows around T_i, T_j, T_k, T_l we construct windows about $T_{i'}, T_{j'}, T_{k'}, T_{l'}$, respectively.

The conclusion of this section is that, by combining a sequence of correctly aligned window of the type (8.1) or (8.2) with a sequence of correctly aligned window of the type (8.3) or (8.4) we obtain a finite sequence of correctly aligned windows that crosses the BZI Λ_k. The shadowing lemma-type of result Lemma 7.1 yields an orbit that visits some prescribed neighborhood in the phase space of each window in Λ. In particular, the shadowing orbit has points that go close to the transition tori.

8.2. **Construction of correctly aligned windows across annular regions separated by invariant tori.** We consider an annular region in Λ between two transition chains of invariant tori. Inside this annular region, we assume the existence of a finite collection of invariant tori that separate the region, as in (A6′-i), and are vertically ordered as in (A6′-ii). Thus, the annular region between the transition chains is not a BZI. We also assume that the scattering map satisfies the non-degeneracy condition (A6′-iii) on these invariant tori. The situation presented in this section is non-generic.

Assume that the annular region in Λ is bounded by a pair of invariant Lipschitz tori T_a and T_b. Let T_a be the end torus of the transition chain of one side, and T_b be the end torus of the transition chain on the other side. We assume that inside the region in the annulus bounded by T_a and T_b there exist a finite collection of invariant tori $\{\Upsilon_h\}_{h=1,\dots,k-1}$. Each Υ_h is either an isolated invariant primary torus, or consists of a hyperbolic periodic orbit together with the upper branches, or with the lower branches, of its stable and unstable manifolds; the stable and unstable manifolds are assumed to coincide. In the second case, there is another invariant torus, say Υ_{h+1}, formed by the remaining branches of the same hyperbolic periodic orbit as for Υ_h. Υ_{h+1} is assumed to be above Υ_h, relative to the I-coordinate, and is assumed to share with Υ_h only the points of the periodic orbit. The region in the annulus bounded by Υ_h and Υ_{h+1} is referred to as a resonant region.

Thus, the region in the annulus between T_a and T_b is divided into a finite number of BZI's and resonant regions.

The constructions in Subsection 8.1 provide a one-sided neighborhood of the type $(S \circ f_{|\Lambda}^{K_a})(D_{iji'j'})$ of some point in T_a, and a one-sided neighborhood $(S^{-1} \circ f_{|\Lambda}^{-K_b})(D_{klk'l'})$ of some point in T_b. Let us denote $D_0 = (S \circ f_{|\Lambda}^{K_a})(D_{iji'j'})$ and $D_{k+1} = (S^{-1} \circ f_{|\Lambda}^{-K_b})(D_{klk'l'})$.

Assume that Υ_1 is an isolated invariant primary torus. We have that $S^{-1}(\Upsilon_1)$ forms with Υ_1 a topological disk D_1 between T_a and Υ_1, which is mapped by S onto a topological disk $S(D_1)$ between Υ_1 and Υ_2. Theorem 6.1 provides us a trajectory that starts from ∂D_0 and ends at ∂D_1. In particular, there exist $K_1 > 0$ and a component of $f^{K_1}(D_0) \cap D_1$ that is a topological disk D_1' whose boundary contains an arc of $S^{-1}(T_1)$. The image of D_1' under S is a one-sided neighborhood $D_1'' \subseteq S(D_1)$ of some point in Υ_1, such that D_1'' is contained in the region between Υ_1 and Υ_2. The boundary of $S(D_1')$ consists of an arc in Υ_1 and an arc in $(S \circ f^{K_1}(\partial D_0))$. See Fig. 12.

Now assume that Υ_1 consists of a hyperbolic periodic orbit, together with the lower branches of its stable and unstable manifolds, and that Υ_2 consists of the same hyperbolic periodic orbit, together with the upper branches of the stable and unstable manifolds. The stable and unstable manifolds are assumed to coincide. Excepting for the common points, Υ_1 is below Υ_2. Thus, Υ_1 and Υ_2 enclose a resonant region within the annulus. We have that $S^{-1}(\Upsilon_1)$ forms with Υ_1 a topological disk D_1 between T_a and Υ_1, which is mapped by S onto a topological disk $S(D_1)$ between Υ_1 and Υ_2. We also have that $S^{-1}(\Upsilon_2)$ forms with Υ_2 a topological disk D_2 between Υ_1 and Υ_2, which is mapped by S onto a topological disk $S(D_2)$ between Υ_2 and Υ_3. By Theorem 6.1 there exist $K_1 > 0$ and a component of $f^{K_1}(D_0) \cap D_1$ that is a topological disk D_1' whose boundary contains an arc of $S^{-1}(\Upsilon_1)$. The image of D_1' under S is a one-sided neighborhood $D_1'' \subseteq S(D_1)$ of some point in Υ_1, contained in the region between Υ_1 and Υ_2. The boundary of

FIGURE 12. Crossing over an isolated invariant primary torus.

FIGURE 13. Construction of windows across a resonance.

D''_1 consists of an arc in Υ_1 and an arc in $S(f^{K_1}(\partial D_0))$. By the argument in the Homoclinic Orbit Theorem (see e.g. [8]), there exists N_1 such that $f^{N_1}(D''_1)$ intersects D_2. Let D'_2 be a component of this intersection that is a topological disk, and whose boundary contains an arc of $S^{-1}(\Upsilon_2)$. Then the image of D'_2 under S is a one sided neighborhood D''_2 of some point in Υ_2 that is contained in the region between Υ_2 and Υ_3. The boundary of D''_2 consists of an arc in Υ_2 and an arc in $(S \circ f^{N_1} \circ S \circ f^{K_1})(\partial D_0)$. See Figure 13.

The main point of this construction is that, using the inner and outer dynamics, one can cross the successive BZI's and resonance regions determined by $\{\Upsilon_h\}_{h=1,\ldots,k}$ one at a time, and obtain at each step a one-sided neighborhood of some point in some Υ_h, which is between Υ_h and Υ_{h+1}, such that the boundary of that neighborhood contains the image of ∂D_0 under a suitable composition of S and powers of f.

This argument can be repeated for each invariant torus Υ_h with $2 \leq h \leq k$, yielding a point $x_a \in \partial D_0$ that is mapped by some appropriate composition of S and powers of f onto a point $x_b \in \partial D_{k+1}$. Then the constructions from Subsection 8.1 can be applied to obtain a window R'_a about ∂D_0, and a window R'_b about ∂D_{k+1}, such that R'_a is correctly aligned with R_b under the appropriate composition of S

and powers of f. Then there exists a shadowing orbit that goes from a neighborhood of R_a in M to a neighborhood of R'_b in M.

If the region between T_a and T_b contains some prescribed collection of Aubry-Mather sets, then Theorem 6.2 yields a shadowing orbit that crosses the region between T_a and T_b and follows the prescribed Aubry-Mather sets as in (2.2).

8.3. Construction of correctly aligned windows along transition chains of tori.
We consider a finite transition chain of invariant primary tori $\{T_1, T_2, \ldots, T_n\}$ in the annulus Λ satisfying (A5). All tori T_k, $k = 1, \ldots, n$, intersect the subset U_+^- of the domain of the scattering map S where S moves points upwards in the annulus Λ. The motion on each torus T_k is topologically transitive. Each torus T_k with $2 \leq k \leq n-1$ is an 'interior' torus, i.e. it can be C^0-approximated from both sides in Λ by other invariant primary tori. For each $k \in \{1, \ldots, n-1\}$, the image $S(T_k)$ of the torus T_k under the scattering map S has a topologically transverse intersection with the torus T_{k+1}. We would like to show that there exists an orbit that visits some ε_k-neighborhood of each torus T_k, $k = 1, \ldots, n$, in the prescribed order. To this end we will construct a sequence of 2-dimensional windows in Λ that are correctly aligned one with another under the scattering map alternatively with some iterates of the inner map, as in Lemma 7.1. Then the lemma will imply the existence of a shadowing orbit to the transition chain.

For each $k \in \{1, \ldots, n-1\}$, we choose and fix a pair of points $x_{k,k+1}^- \in T_k$ and $x_{k,k+1}^+ \in T_{k+1}$ such that $S(x_{k,k+1}^-) = x_{k,k+1}^+$ and $S(T_k)$ intersects T_{k+1} transversally at $x_{k,k+1}^+$.

We construct inductively a sequence of correctly aligned windows in Λ along the tori $\{T_1, T_2, \ldots, T_n\}$, such that each window is correctly aligned with the next window in the sequence either by the outer map or by some sufficiently large power of the inner map. Moreover, each window will be contained in some ε-neighborhood of some transition torus. We start the inductive construction at T_1. Consider the point $x_{1,2}^- \in T_1$ with $S(x_{1,2}^-) = x_{1,2}^+ \in T_2$ as above.

We construct a window R'_1 about T_1 as follows. Let $\bar{e}'_1$ be an arc contained in T_1, and $\chi'_1 : \mathbb{R}^2 \to \Lambda$ a C^0-local parametrization such that $\chi'_1([0,1] \times \{0\}) = \bar{e}'_1$. Then we define

$$R'_1 = \chi'_1([0,1] \times [-\delta'_1, \delta'_1]),$$
$$R'^{\text{exit}}_1 = \chi'_1(\partial[0,1] \times [-\delta'_1, \delta'_1]),$$

where $0 < \delta'_1 < \varepsilon_1$. We choose $\bar{e}'_1$ and δ'_1 sufficiently small so that and $S(\bar{e}'_1)$ intersects T_2 only at $x_{1,2}^+$, and also so that R'_1 defined as above is contained in U_+^-. The exit direction of R'_1 is in the direction of the torus T_1.

The image $S(R'_1)$ of R'_1 under the scattering map is a topological rectangle. Since $S(\bar{e}'_1)$ is transverse to T_2 at $x_{1,2}^+$, then by choosing δ'_1 sufficiently small we ensure that the components of $S(R'^{\text{exit}}_1)$ lie on opposite sides of T_2. Thus, the exit direction of $S(R'_1)$ is across the torus T_2.

Next we consider the point $x_{2,3}^- \in T_2$ with $S(x_{2,3}^-) = x_{2,3}^+ \in T_3$ as above. We construct a window R'_2 about T_2 in a manner similar to R'_1:

$$R'_2 = \chi'_2([0,1] \times [-\delta'_2, \delta'_2]),$$
$$R'^{\text{exit}}_2 = \chi'_2(\partial[0,1] \times [-\delta'_2, \delta'_2]),$$
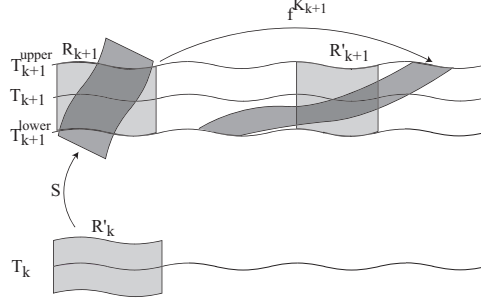
FIGURE 14. Construction of windows along a transition chain.

where $\bar{e}_2'$ is an arc contained in T_2, $\chi_2' : \mathbb{R}^2 \to \Lambda$ is a C^0-local parametrization such that $\chi_2'([0,1] \times \{0\}) = \bar{e}_2'$, and $0 < \delta_2' < \varepsilon_2$ is chosen sufficiently small.

Now we construct a window R_2 about the point $x_{1,2}^+$ such that R_1' is correctly aligned with R_2 under S and R_2 is correctly aligned with R_2' under some power of $f_{|\Lambda}$. We choose an arc $\bar{e}_2$ in T_2 containing $x_{1,2}^+$ such that $\bar{e}_2 \supseteq S(R_1') \cap T_2$. We choose a pair of invariant tori T_2^{lower} and T_2^{upper} such that $T_2^{\text{lower}} \prec T_2 \prec T_2^{\text{upper}}$ and the exit set components of $S(R_1')$, as well as the entry set components of R_2', are outside of the annulus bounded by T_2^{lower} and T_2^{upper}, on the both sides of the annulus. Furthermore, we require that T_2^{lower} and T_2^{upper} should be ε_2-close to T_2 in the C^0-topology. The existence of such neighboring tori T_2^{lower} and T_2^{upper} to T_2 is guaranteed by (A5-iv). Let $\chi_2 : \mathbb{R}^2 \to \Lambda$ be a C^0-local parametrization such that $\chi_2(\{0\} \times [0,1]) = \bar{e}_2$, $\chi_2(\{-\delta_2\} \times [0,1]) \subseteq T_2^{\text{lower}}$ and $\chi_2(\{\delta_2\} \times [0,1]) \subseteq T_2^{\text{upper}}$, for some $0 < \delta_2 < \varepsilon_2$ sufficiently small. Define

$$R_2 = \chi_2([-\delta_2, \delta_2] \times [0,1]),$$
$$R_2^{\text{exit}} = \chi_2(\partial[-\delta_2, \delta_2] \times [0,1]).$$

The exit set components of R_2 lie on the invariant tori $T_2^{\text{lower}}, T_2^{\text{upper}}$ neighboring T_2.

Since the motion on the torus T_2 is topological transitive, there exists a power $f_{|\Lambda}^{K_2}$ of $f_{|\Lambda}$ such that R_2 is correctly aligned with R_2' under $f_{|\Lambda}^{K_2}$. See Figure 14. We have obtained the windows R_1' about T_1 and R_2', R_2 about T_2 such that R_1' is correctly aligned with R_2 under S and R_2 is correctly aligned with R_2' under some power of $f_{|\Lambda}$. This ends the initial step of the inductive construction.

The inductive step goes on similarly, yielding the windows R_k' about T_k and R_{k+1}', R_{k+1} about T_{k+1} such that R_k' is correctly aligned with R_{k+1} under S and R_{k+1} is correctly aligned with R_{k+1}' under some power of $f_{|\Lambda}$. See Figure 14.

The construction proceeds inductively until a sequence of windows in Λ of the following type is obtained

$$R_1', R_2, R_2', \ldots, R_k, R_k', R_{k+1}, R_{k+1}', \ldots, R_n,$$

where for each $k \in \{1, \ldots, n-1\}$ we have that R_k' is correctly aligned with R_{k+1} under S, and R_{k+1} is correctly aligned with R_{k+1}' under $f_{|\Lambda}^{K_{k+1}}$ for some $K_{k+1} > 0$. Each window R_k, R_k' is contained in an ε_k-neighborhood of the torus T_k. Applying

the shadowing lemma-type of result Lemma 7.1 provides an orbit that visits an ε_k-neighborhood in the phase space of each window in the sequence, and in particular of each torus in the transition chain.

8.4. **Gluing correctly aligned windows across BZI's with correctly aligned windows along transition chains of tori.** We consider three transition chains of tori $\{T_i\}_{i=i_{k-1}+1,\ldots,i_k}$, $\{T_i\}_{i=i_k+1,\ldots,i_{k+1}}$, $\{T_i\}_{i=i_{k+1}+1,\ldots,i_{k+2}}$, with the property that each one of the regions between T_{i_k} and T_{i_k+1}, and between $T_{i_{k+1}}$ and $T_{i_{k+1}+1}$, is either a BZI as in (A6), or it contains a finite number of invariant tori that separate the region as in (A6'). Inside each region there is a prescribed collection of Aubry-Mather sets as in (A7)

The constructions in Subsection 8.1 and in Subsection 8.2 yield correctly aligned windows in Λ that cross the region between T_{i_k} and T_{i_k+1}, and correctly aligned windows in Λ that cross the region between $T_{i_{k+1}}$ and $T_{i_{k+1}+1}$. The construction in Subsection 8.3 yield sequences of correctly aligned windows along the adjacent transition chains. The choices of the windows constructed along the transition chains depend on the choices of the windows that cross the region between T_{i_k} and T_{i_k+1}, and of the windows that cross the region between $T_{i_{k+1}}$ and $T_{i_{k+1}+1}$. Propagating the construction of windows starting from T_{i_k+1} and moving forward along the transition chain $\{T_i\}_{i=i_k+1,\ldots,i_{k+1}}$, and the construction of windows starting from $T_{i_{k+1}}$ and moving backward along the same transition chain $\{T_i\}_{i=i_k+1,\ldots,i_{k+1}}$, may result in a pair of windows about some intermediate torus that are not correctly aligned. We would like to glue this sequences of windows in a manner that is correctly aligned, without having to revise the windows constructed to that point.

Assume that T_j is one of the tori $\{T_i\}_{i=i_k+1,\ldots,i_{k+1}}$, with $j \in \{i_k+2,\ldots,i_{k+1}-1\}$. Assume that one has already constructed a window R_j about T_j by propagating the construction from T_{i_k+1} and moving forward along the transition chain, and another window R'_j about T_j by propagating the construction from $T_{i_{k+1}}$ and moving backwards along the transition chain. The window R_j is of the form

$$R_j = \chi_j([-\delta_j, \delta_j] \times [0,1]),$$
$$R_j^{\text{exit}} = \chi_j(\partial[-\delta_j, \delta_j] \times [0,1]),$$

where $\chi_j : \mathbb{R}^2 \to \Lambda$ is a C^0-local parametrization with $\chi_j(\{0\} \times [0,1]) \subseteq T_j$, and $\chi_j(\{-\delta_j\} \times [0,1]) \subseteq T_j^{\text{lower}}$ and $\chi_j(\{\delta_j\} \times [0,1]) \subseteq T_j^{\text{upper}}$, where T_j^{lower} and T_j^{upper} are two primary invariant tori on the opposite sides of T_j. The window R'_j is of the form

$$R'_j = \chi'_j([0,1] \times [-\delta'_j, \delta'_j]),$$
$$R'^{\text{exit}}_j = \chi'_j(\partial[0,1] \times [-\delta'_j, \delta'_j]),$$

where $\chi'_j : \mathbb{R}^2 \to \Lambda$ is a C^0-local parametrization with $\chi'_j([0,1] \times \{0\}) \subseteq T_j$, and $\chi'_j([0,1] \times \{-\delta'_j\}) \subseteq T'^{\text{lower}}_j$ and $\chi'_j([0,1] \times \{\delta'_j\}) \subseteq T'^{\text{upper}}_j$, where T'^{lower}_j and T'^{upper}_j are two primary invariant tori on the opposite sides of T_j.

Let us assume that the annular region between T'^{lower}_j and T'^{upper}_j is inside the region between T_j^{lower} and T_j^{upper}. We construct a new window R''_j about T_j, such that R_j is correctly aligned with R''_j under the identity map, and R''_j is correctly
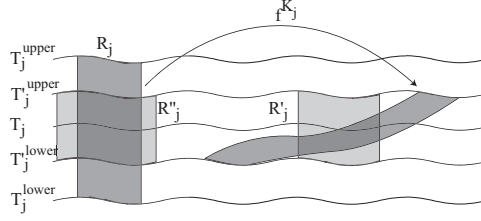
FIGURE 15. Gluing sequences of correctly aligned windows along a transition chain.

aligned with R'_j under some power of f. We let R''_j is of the form

$$R''_j = \chi''_j([-\delta''_j, \delta''_j] \times [0,1]),$$
$$R''^{\text{exit}}_j = \chi''_j(\partial[-\delta''_j, \delta''_j] \times [0,1]),$$

where $\chi''_j : \mathbb{R}^2 \to \Lambda$ is a C^0-local parametrization with $\chi''_j(\{0\} \times [0,1]) \supseteq \chi'_j(\{0\} \times [0,1])$, and $\chi''_j(\{-\delta''_j\} \times [0,1]) \subseteq T'^{\text{lower}}_j$ and $\chi''_j(\{\delta''_j\} \times [0,1]) \subseteq T'^{\text{upper}}_j$, for some $\delta''_j > 0$. Since the motion on the torus T_j is topological transitive, there exists a power $f^{K_j}_{|\Lambda}$ of $f_{|\Lambda}$ such that R''_j is correctly aligned with R'_j under $f^{K_j}_{|\Lambda}$. See Figure 15. We have obtained that R_j is correctly aligned with R''_j under the identity map, and R''_j is correctly aligned with R'_j under some power $f^{K_j}_{|\Lambda}$ of $f_{|\Lambda}$.

The case when the annular region between T^{lower}_j and T^{upper}_j is inside the region between T'^{lower}_j and T'^{upper}_j results in the window R_j correctly aligned with R'_j under some power of f, without the need of creating an intermediate window R''_j. The case when neither annular region is contained in the other annular region can be dealt similarly by constructing an intermediate window R''_j.

Through this process, a sequence of correctly windows constructed forward along a transition chain can be glued, in a correctly aligned manner, with a sequence of correctly windows constructed backwards along the same transition chain.

8.5. **Proof of Theorem 2.1.** To summarize, in Subsection 8.1 and in Subsection 8.2, we described the construction of correctly aligned windows that cross an annular regions between two consecutive transition chains of invariant tori. These annular regions are BZI's or else they are separated by some finite collection of invariant tori. If each annular region has some prescribed collection od Aubry-Mather sets, the construction yields windows that follow these Aubry-Mather sets. In Subsection 8.3 yield sequences of correctly aligned windows each transition chain. In Subsection 8.4 the construction of a sequence of correctly windows constructed forward along a transition chain can be glued, in a correctly aligned manner, with a the construction of a sequence of correctly windows constructed backwards along the same transition chain. Thus, starting from some initial annular region, one can construct sequences of correctly aligned windows, forward and backwards, along infinitely many topological transition chains interspersed with annular regions. The Shadowing Lemma-type of result Theorem 7.1 implies the existence of an orbit that gets ε_i-close to each transition torus T_i, and also follows each Aubry-Mather set Σ_i for a prescribed time n_i.

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